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Bi-intuitionistic Boolean Bunched Logic

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Abstract

We formulate and investigate a *bi-intuitionistic* extension, BiBBI, of the well known bunched logic Boolean BI (BBI), obtained by combining classical logic with *full intuitionistic* linear logic as considered by Hyland and de Paiva (as opposed to standard multiplicative intuitionistic linear logic). Thus, in addition to the multiplicative conjunction $*$ with its adjoint implication $-\!\!*$ and unit $\top^*$, which are provided by BBI, our logic also features an intuitionistic multiplicative *disjunction* $\overset{*}{\vee}$, with its adjoint co-implication $\setminus^*$ and unit $\bot^*$. "Intuitionism" for the multiplicatives means here that disjunction and conjunction are related by a *weak distribution* principle, rather than by De Morgan equivalence.

We formulate a Kripke semantics for BiBBI in which all the above multiplicatives are given an intuitionistic reading in terms of resource operations. Our main theoretical result is that validity according to this semantics exactly coincides with provability in our logic, given by a standard Hilbert-style axiomatic proof system. In particular, we isolate the Kripke frame conditions corresponding to various natural logical principles of FILL, which allows us to present soundness and completeness results that are modular with respect to the inclusion or otherwise of these axioms in the logic. Completeness follows by embedding BiBBI into a suitable modal logic and employing the famous Sahlqvist completeness theorem.

We also investigate the Kripke models of BiBBI in some detail, chiefly in the hope that BiBBI might be used (like BBI) to underpin program verification applications based on *separation logic*. Interestingly, it turns out that the heap-like memory models of separation logic are also models of BiBBI, in which disjunction can be interpreted using a natural notion of *heap intersection*.

1 Introduction

Bunched logics, which can be understood as an orthogonal combination of some variant of standard propositional logic with some variant of multiplicative linear logic [3, 23], have applications in computer science as a means of expressing and manipulating properties of *resource* [21, 24, 4]. Most notably, *separation logic* [25], which has been successfully employed in large-scale program verification [7, 26, 13] is based upon the bunched logic Boolean BI (from now on BBI) obtained by combining ordinary classical logic with *multiplicative intuitionistic linear logic* (from now on MILL) [12].

BBI has a very simple and appealing Kripke frame semantics: a model of BBI is simply a certain type of (relational) commutative monoid, and a formula can then be read directly in such a model as a subset of its elements, typically understood as abstract *resources*. The classical connectives have their usual meanings, and the multiplicative MILL connectives (called *multiplicative*) are given "resource composition" readings: A multiplicative conjunction of formulas $A * B$ denotes those elements which divide, via the monoid operation, into two elements satisfying A and B respectively. (In separation logic, $*$ expresses the division of heap memories into two disjoint pieces.) The unit $\top^*$ of $*$ denotes the set of units of the monoid, and an implication (or "magic wand") $A \mathrel{-\!\!*} B$ denotes those elements that, when extended with an element satisfying A, always yield an element satisfying B.

For some time following the inception of bunched logic it was unclear whether multiplicative analogues of other standard logical connectives — particularly disjunction and negation — could be given a similarly intuitive (and non-trivial) resource interpretation. A first positive answer to this question came in the form of a complete frame semantics for *Classical BI* (CBI) [4], given by extending classical logic with multiplicative (classical) linear logic (MLL) rather than MILL. Thus in CBI, as in MLL, the multiplicative connectives are related by the expected de Morgan equivalences. These strong proof-theoretic equivalences have a model-theoretic correlate: CBI-models are BBI-models in which every element of the monoid has a unique "dual" (in a certain technical sense). It is easy to find BBI-models in which such duals do not exist, and in fact CBI is nonconservative over BBI. In particular, the heap-like models employed in separation logic are not models of CBI, which unfortunately rules out using the convenient logical symmetries of CBI to reason about separation logic over these models. This leaves open the question of whether there might be bunched logics sitting in between BBI and CBI in which multiplicative disjunction, negation etc. are interpretable but do not obey the de Morgan laws of MLL. Additionally, one could ask whether any putative such logic might be used to reason about heap-like models, and thus might have potential applications to separation logic.

In this paper, we give positive answers to both of the aforementioned questions by formulating a so-called *bi-intuitionistic* version of BBI, based on combining classical logic with *full intuitionistic* linear logic (FILL) as considered in [16]. FILL adds the linear disjunction ("par") and its unit to MILL; in BiBBI, we write these connectives as $\overset{*}{\vee}$ and $\bot^*$ respectively. The disjunction $\overset{*}{\vee}$ (which the reader is invited to read as "mor") also has a natural adjoint *co-implication* which we write as $\setminus^*$ (and refer to as "magic slash"). This adjoint was not present in the original formulation of FILL, but has been considered recently by Clouston et al. in order to formulate a display calculus for FILL with the cut-elimination property [9]. Here, we include $\setminus^*$ partly for the sake of symmetry, but also because, like in [9], it plays a useful rôle in some of our technical developments.

Provability in BiBBI can be given in the usual way simply by combining suitable Hilbert systems for classical logic and for FILL. One of our main contributions in this paper is to formulate a suitable Kripke frame semantics for BiBBI, and to show that provability is *sound and complete* with respect to validity in this semantics. Soundness is an easy result, whereas completeness follows by embedding BiBBI into a suitable modal logic and deploying Sahlqvist's well-known completeness theorem for modal logic (see, e.g., [2]). BiBBI is easily seen to be "in between" the logics BBI and CBI, in the sense that BiBBI is an extension of BBI and CBI an extension of BiBBI.

We consider a number of variants of BiBBI, based on whether or not various natural logical principles of FILL are included in our logic. For each such principle, we identify a corresponding first-order frame condition on the Kripke models of BiBBI which exactly defines the validity of the principle in any such model (see Table 1 in Section 3). As a helpful side effect, this enables us to present our soundness and completeness results so as to apply to any variant of BiBBI. Probably the most interesting principle of FILL is the so-called *weak distribution* of multiplicative conjunction over disjunction, given by

$$A * (B \mathbin{\overset{*}{\vee}} C) \vdash (A * B) \mathbin{\overset{*}{\vee}} C.$$

This axiom is of crucial importance in FILL, and consequently in BiBBI, because it provides the *only* connection between the $(*, -\!\!*, \top^*)$ and $(\overset{*}{\vee}, \backslash^*, \bot^*)$ fragments of the logic(s). The frame condition corresponding to the above weak distribution law is surprisingly complicated, and in Section 4 we undertake a more detailed investigation of the models of BiBBI obeying this condition. In particular, we find that the heap-like models of separation logic can be extended to models of BiBBI obeying the weak distribution condition if we interpret the multiplicative disjunction $\overset{*}{\vee}$ using certain natural notions of *intersection of heaps*. We also present some general techniques for constructing BiBBI-models obeying weak distribution.

The remainder of this paper is structured as follows. In Section 2 we recall the model-theoretic and proof-theoretic characterisations of BBI and CBI. We then introduce our biintuitionistic bunched logic BiBBI, via both a Kripke frame semantics and a Hilbert-style axiomatic proof system, in Section 3. In Section 4 we investigate the Kripke models of BiBBI in more detail, and present some general constructions for BiBBI-models obeying the weak distribution law. Section 5 gives our completeness proof, and Section 6 concludes.

2 BBI and CBI: an overview

In this section, we recall the basic characterisations of *validity* (based on Kripke frame semantics) and *provability* in the bunched logics BBI [17, 11] and CBI [4].

We assume a denumerably infinite set $\mathcal{V}$ of propositional variables, and write $\mathcal{P}(X)$ for the powerset of a set X.

2.1 Boolean BI

Here we briefly recall the syntax, proof theory, and semantics of BBI, as can be found in several places in the literature. We remark that the metatheory of BBI has been quite extensively studied from the point of view of completeness [11, 15], expressivity [18, 6], decidability [5, 19], and proof theory [3, 22, 20].

Definition 2.1. BBI-*formulas* are built from propositional variables $P \in \mathcal{V}$ using the standard formula connectives $\top, \bot, \neg, \wedge, \vee, \rightarrow$ of propositional classical logic, and the so-called "multiplicative" connectives, consisting of the constant $\top^*$ and binary operators $*$ and $-\!\!*$.

By convention, $\neg$ has the highest precedence, followed by $*$, $\wedge$ and $\vee$, with $\rightarrow$ and $-\!\!*$ having lowest precedence.

Definition 2.2. *Provability* in BBI is given by extending a complete Hilbert system for classical logic with the following axioms and inference rules for $*$, $-\!\!*$ and $\top^*$. The "sequent" notation $A \vdash B$ is used as syntactic sugar for the formula $A \to B$.

$$A * B \vdash B * A \qquad A \vdash A * \top^* \qquad A * \top^* \vdash A \qquad A * (B * C) \vdash (A * B) * C$$

$$\frac{A_1 \vdash B_1 \quad A_2 \vdash B_2}{A_1 * A_2 \vdash B_1 * B_2} \qquad \frac{A * B \vdash C}{A \vdash B -\!\!* C} \qquad \frac{A \vdash B -\!\!* C}{A * B \vdash C}$$

Definition 2.3. A BBI-*frame* is a a tuple $\langle W, \circ, E \rangle$, where W is a set (of "worlds"), $\circ : W \times W \to \mathcal{P}(W)$ and $E \subseteq W$. We extend $\circ$ pointwise to $\mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$ by

$$W_1 \circ W_2 \;=_{\text{def}}\; \bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 \circ w_2 \ .$$

A BBI-frame $\langle W, \circ, E \rangle$ is a BBI-*model* if $\circ$ is commutative and associative, and $w \circ E = \{w\}$ for all $w \in W$. (By definition, the latter means that $\bigcup_{e \in E} w \circ e = \{w\}$ for all $w \in W$.) We call E the set of *units* of the model $\langle W, \circ, E \rangle$.

If the binary operation $\circ$ in a BBI-model $M = \langle W, \circ, E \rangle$ satisfies $|w_1 \circ w_2| \leq 1$ for all $w_1, w_2 \in W$, then we say that M is *partial functional* and understand $\circ$ as a partial function of type $W \times W \rightharpoonup W$.

Example 2.4. The standard *heap model* $\langle \text{Heaps}, \circ, \{e\} \rangle$ of separation logic [25] is defined as follows. First, $\text{Heaps} = \text{Loc} \rightharpoonup_{\text{fin}} \text{Val}$ is the set of partial functions mapping finitely many locations Loc to values Val. We write $\mathsf{dom}(h)$ for the *domain* of heap h, i.e. the set of locations on which h is defined. We define $h_1 \circ h_2$ to be the union of heaps h_1 and h_2 if $\mathsf{dom}(h_1)$ and $\mathsf{dom}(h_2)$ are disjoint (and undefined otherwise), and we let e be the *empty heap* with $\mathsf{dom}(e) = \emptyset$. It is straightforward to verify that $\langle \text{Heaps}, \circ, \{e\} \rangle$ is a (partial functional) BBI-model.

Definition 2.5. Let $M = \langle W, \circ, E \rangle$ be a BBI-model. A *valuation* for M is a function ρ that assigns to each atomic proposition P a set $\rho(P) \subseteq W$. Given any valuation ρ for M, any $w \in W$ and any $\mathcal{L}$-formula A, we define the forcing relation $w \models_\rho A$ by induction on A:

$$\begin{array}{rcl}
w \models_\rho P & \Leftrightarrow & w \in \rho(P) \\
w \models_\rho \top & \Leftrightarrow & \text{always} \\
w \models_\rho \bot & \Leftrightarrow & \text{never} \\
w \models_\rho \neg A & \Leftrightarrow & w \not\models_\rho A \\
w \models_\rho A_1 \wedge A_2 & \Leftrightarrow & w \models_\rho A_1 \text{ and } w \models_\rho A_2 \\
w \models_\rho A_1 \vee A_2 & \Leftrightarrow & w \models_\rho A_1 \text{ or } w \models_\rho A_2 \\
w \models_\rho A_1 \to A_2 & \Leftrightarrow & w \models_\rho A_1 \text{ implies } w \models_\rho A_2 \\
w \models_\rho \top^* & \Leftrightarrow & w \in E \\
w \models_\rho A_1 * A_2 & \Leftrightarrow & \exists w_1, w_2 \in W.\ w \in w_1 \circ w_2 \text{ and } w_1 \models_\rho A_1 \text{ and } w_2 \models_\rho A_2 \\
w \models_\rho A_1 -\!\!* A_2 & \Leftrightarrow & \forall w', w'' \in W.\ \text{if } w'' \in w \circ w' \text{ and } w' \models_\rho A_1 \text{ then } w'' \models_\rho A_2
\end{array}$$

A is said to be *valid in* M if $w \models_\rho A$ for any valuation ρ and for all $w \in W$, and BBI-*valid* if it is valid in all BBI-models.

Theorem 2.6 (Soundness / completeness of BBI [11])**.** *A* BBI-*formula is* BBI-*valid if and only if it is* BBI-*provable.*

2.2 Classical BI

Here we recall the definitions of provability and validity in the bunched logic CBI [4].

Definition 2.7. CBI-*formulas* are defined as BBI-formulas (Defn. 2.1), except that they may also contain the "multiplicative falsum" constant $\bot^*$. We write $\sim A$ as an abbreviation for $A \mathrel{-\!\!*} \bot^*$, and $A \mathbin{\check{\vee}}^{*} B$ as an abbreviation for $\sim(\sim A * \sim B)$.

Definition 2.8. Provability in CBI is defined as for BBI (Defn. 2.2) except that the following axiom is also included:

$$\sim\sim A \vdash A$$

Definition 2.9. A CBI-*model* is given by a tuple $\langle W, \circ, E, U \rangle$, where $\langle W, \circ, E \rangle$ is a BBI-model (see Defn. 2.3), $U \subseteq W$, and for each $w \in W$, there is a unique $-w \in W$ (the "dual" of w) satisfying $(w \circ -w) \cap U \neq \emptyset$.

Definition 2.10. A valuation for a CBI-model and satisfaction $w \models_\rho A$ of a CBI-formula A by the world w and valuation ρ are defined as for BBI (Defn. 2.5), except that we add the following clause for satisfaction of the multiplicative falsum:

$$w \models_\rho \bot^* \quad \Leftrightarrow \quad w \notin U$$

Example 2.11 ([4])**.** Let $\{0,1\}^\omega$ denote all infinite strings over the alphabet $\{0,1\}$ of "bits". Writing 0^ω for the infinite string of 0s and XOR for the standard bitwise exclusive-or operation on infinite bit strings, it is easy to check that $\langle \{0,1\}^\omega, \mathrm{XOR}, \{0^\omega\} \rangle$ is a BBI-model.

Now, for any $\sigma \in \{0,1\}^\omega$, let $\overline{\sigma}$ be the bit string obtained from σ by flipping each of its bits from 0 to 1 and vice versa (so e.g. $\overline{0^\omega} = 1^\omega$). For any $\sigma \in \{0,1\}^\omega$, the string $\overline{\sigma}$ is the unique $-\sigma \in \{0,1\}^\omega$ such that $\sigma \ \mathrm{XOR} \ {-\sigma} = 1^\omega$. Thus $\langle \{0,1\}^\omega, \mathrm{XOR}, \{0^\omega\}, \{1^\omega\} \rangle$ is a CBI-model.

Proposition 2.12. *The heap model* $\langle \mathrm{Heaps}, \circ, \{e\} \rangle$ *of* BBI *defined in Example 2.4 is not a* CBI*-model. That is, there is no set* $U \subseteq \mathrm{Heaps}$ *such that* $\langle \mathrm{Heaps}, \circ, \{e\}, U \rangle$ *is a* CBI*-model.*

Proof. Suppose for contradiction that such a U exists. Clearly U must be nonempty. In fact $|U| = 1$, for if $u_1, u_2 \in U$ then $e \circ u_1 = \{u_1\} \subseteq U$ and $e \circ u_2 = \{u_2\} \subseteq U$, and by the CBI axiom we then have $u_1 = -e = u_2$. So $U = \{u\}$, say. Note that $u \in \mathrm{Heaps}$ and thus $\mathsf{dom}(u)$ is finite. Let h be a heap with $\mathsf{dom}(h) \supset \mathsf{dom}(u)$ (there are infinitely many such h). Then there exists a heap $-h$ such that $h \circ -h = u$ by the CBI-axiom, but it is clear that there is no such heap. □

Given a CBI-model $\langle W, \circ, E, U \rangle$, the condition in Definition 2.9 induces a function $- : W \to W$ sending w to $-w$, and this function is necessarily involutive, i.e. $--w = w$ for any $w \in W$ (see [4])[1]. Moreover, it is easy to show that $-E = U$.

Using this definition, and extending $-$ pointwise to sets in a similar way to $\circ$ (see Defn. 2.3), we obtain the following clauses for satisfaction of multiplicative negation $\sim$ and disjunction $\mathbin{\check{\vee}}^{*}$:

$$\begin{array}{rcl} w \models_\rho \sim A & \Leftrightarrow & -w \not\models_\rho A \\ w \models_\rho A \mathbin{\check{\vee}}^{*} B & \Leftrightarrow & \forall w_1, w_2 \in W.\ \text{if } w \in -(-w_1 \circ -w_2) \text{ then } w_1 \models_\rho A \text{ or } w_2 \models_\rho B \end{array}$$

Theorem 2.13 (Soundness / completeness of CBI [4, 3])**.** *A* CBI*-formula is* CBI*-valid if and only if it is* CBI*-provable.*

Proof. It is shown in [4] that a display calculus for CBI is sound and complete for CBI-validity, and in [3] that provability in this display calculus is equivalent to provability in the minimal system we present here. (In [3], a particular presentation of classical logic is chosen, but clearly any sound and complete presentation suffices.) □

[1] In [4] the function $- : W \to W$ is given as part of the model, but in fact U determines $-$ (and vice versa).

Theorem 2.14 ([4])**.** CBI *is a non-conservative extension of* BBI. *That is, there are* BBI-*formulas that are* CBI-*valid but not* BBI-*valid.*

3 BiBBI: Bi-intuitionistic Boolean bunched logic

In this section we introduce our bi-intuitionistic Boolean bunched logic, BiBBI. This logic extends standard BBI with the multiplicative disjunction $\overset{*}{\vee}$, together with its adjoint multiplicative co-implication $\backslash^{*}$ (a.k.a. "magic slash") and the multiplicative falsum $\bot^*$. These connectives are here given an essentially *intuitionistic* interpetation, in analogy to their readings in FILL [16]; in particular, $*$ and $\overset{*}{\vee}$ are not connected by de Morgan equivalences.

Below, we define a suitable notion of a Kripke model for BiBBI, and then set out the interpretation of formulas in these models, along similar lines to the semantics of BBI in CBI in Section 2. Our choice of models and interpretation is designed to achieve several complementary objectives:

1. BiBBI is an *extension* of BBI (i.e., for BBI-formulas, validity in BBI implies validity in BiBBI). Furthermore, when a suitable "classicality" axiom is added to BiBBI, it collapses into CBI (see Proposition 3.7). Thus, BiBBI can be seen as an intermediate logic between BBI and CBI.

2. Our interpretation of multiplicative disjunction $\overset{*}{\vee}$ in BiBBI is dual to the interpretation of multiplicative conjunction $*$, in the sense that $\overset{*}{\vee}$ can be read as a binary *box modality* in modal logic [2], while $*$ can be read as a binary *diamond modality.*

3. For each natural logical principle governing the behaviour of $\overset{*}{\vee}$, $\backslash^{*}$ and $\bot^*$ (drawn from the axioms of FILL), one can write down an equivalent first-order condition on BiBBI-models (see Table 1).

4. Finally, for *any* variant of BiBBI obtained by taking some combination of logical axioms from Table 1, we achieve a suitable soundness and completeness result for that variant with respect to the associated class of models.

Definition 3.1. A BiBBI-*formula* is defined as a BBI-formula (Defn. 2.1), except that it may also contain the multiplicative constant $\bot^*$, and the binary multiplicative connectives $\backslash^{*}$ and $\overset{*}{\vee}$. That is, BiBBI features the following multiplicative connectives:

$$\top^*, *, -\!\!*, \bot^*, \overset{*}{\vee}, \backslash^{*} \ .$$

As in CBI, we write $\sim\! A$ as an abbreviation for $A -\!\!* \bot^*$.

Next, we present a basic characterisation of validity for BiBBI-formulas and an associated notion of basic provability. Then, we extend these characterisations to deal with various further logical properties, which we regard as a sort of "logical buffet" from which we can obtain a logic by choosing the principles we wish to include. However, we include commutativity of $\overset{*}{\vee}$ as a basic principle for technical convenience, since a non-commutative $\overset{*}{\vee}$ naturally leads to both $\backslash^{*}$ and $\bot^*$ splitting into two connectives (acting on the left and right of $\overset{*}{\vee}$).

Definition 3.2. A *basic* BiBBI-*model* is given by $\langle W, \circ, E, \triangledown, U\rangle$, where $\langle W, \circ, E\rangle$ is a BBI-model, $U \subseteq W$ and $\triangledown\colon W \times W \to \mathcal{P}(W)$ is commutative. We extend $\triangledown$ pointwise to sets in a similar manner to $\circ$:

$$W_1 \triangledown W_2 \ =_{\text{def}} \ \textstyle\bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 \triangledown w_2 \ .$$

A valuation for a basic BiBBI-model $M = \langle W, \circ, E, \triangledown, U \rangle$ is defined as in Definition 2.5. Satisfaction $w \models_\rho A$ of a BiBBI-formula A by the valuation ρ and world w is given by extending the forcing relation in Definition 2.5 with the following clauses for $\bot^*$, $\overset{*}{\vee}$ and $\backslash^*$:

$$\begin{array}{rcl} w \models_\rho \bot^* & \Leftrightarrow & w \notin U \\ w \models_\rho A \overset{*}{\vee} B & \Leftrightarrow & \forall w_1, w_2 \in W.\ w \in w_1 \triangledown w_2 \text{ implies } w_1 \models_\rho A \text{ or } w_2 \models_\rho B \\ w \models_\rho A \mathbin{\backslash^*} B & \Leftrightarrow & \exists w', w'' \in W.\ w'' \in w' \triangledown w \text{ and } w'' \models_\rho A \text{ and } w' \not\models_\rho B \end{array}$$

Similarly to BBI and CBI (see Section 2), a BiBBI-formula A is *valid in* M if $w \models_\rho A$ for all $w \in W$, and BiBBI-*valid* if it is valid in all BiBBI-models.

Definition 3.3. *Basic provability* in BiBBI is given by extending the proof system for BBI (see Definition 2.2) with the following axioms and inference rules:

Monotonicity:	Residuation:		Commutativity:
$\dfrac{A_1 \vdash B_1 \quad A_2 \vdash B_2}{A_1 \overset{*}{\vee} A_2 \vdash B_1 \overset{*}{\vee} B_2}$	$\dfrac{A \vdash B \overset{*}{\vee} C}{A \mathbin{\backslash^*} B \vdash C}$	$\dfrac{A \mathbin{\backslash^*} B \vdash C}{A \vdash B \overset{*}{\vee} C}$	$A \overset{*}{\vee} B \vdash B \overset{*}{\vee} A$

Theorem 3.4 (Basic soundness)**.** *If a formula A is provable in the system for basic* BiBBI *(Definition 3.3) then it is valid in all basic* BiBBI*-models.*

Proof. By soundness for standard BBI (Theorem 2.6) it suffices to show that the axioms and rules in Definition 3.3 preserve validity in an arbitrary basic BiBBI-model $\langle W, \circ, E, \triangledown, U \rangle$. We distinguish a case for each rule.

Monotonicity: Assume that the premises $A_1 \vdash B_1$ and $A_2 \vdash B_2$ are valid. Then, assuming $w \models_\rho A_1 \overset{*}{\vee} A_2$, we must show that $w \models_\rho B_1 \vee B_2$, i.e. that $w \in w_1 \triangledown w_2$ implies $w_1 \models_\rho B_1$ or $w_2 \models_\rho B_2$. Assume $w \in w_1 \triangledown w_2$. Since $w \models_\rho A_1 \overset{*}{\vee} A_2$, either $w \models_\rho A_1$ or $w \models_\rho A_2$. Using validity of the two premises of the rule, either $w \models_\rho B_1$ or $w \models_\rho B_2$, as required.

Residuation: We just show soundness of one of the rules here; the other is similar. Assume that the premise $A \mathbin{\backslash^*} B \vdash C$ is valid. Then, assuming $w \models_\rho A$, we must show $w \models_\rho B \overset{*}{\vee} C$. That is, assuming $w \in w_1 \triangledown w_2$, we must show either $w_1 \models_\rho B$ or $w_2 \models_\rho C$. Thus, we assume $w_1 \not\models_\rho B$ and must show $w_2 \models_\rho C$. Since, collecting assumptions, we have $w \in w_1 \triangledown w_2$ with $w \models_\rho A$ and $w_1 \not\models_\rho B$, it holds that $w_2 \models_\rho A \mathbin{\backslash^*} B$, whence $w_2 \models_\rho C$ as required by the validity of the rule premise.

Commutativity: Follows immediately from commutativity of $\triangledown$. □

Definition 3.5. A *variant* of BiBBI is obtained by adding, for any combination of "principles" from Table 1, (a) the logical axiom $\mathcal{A}$ for that principle to the basic BiBBI proof system in Definition 3.3, and (b) the frame condition $\mathcal{F}(\mathcal{A})$ for that principle as an additional condition on the basic BiBBI-models in Definition 3.2.

Proposition 3.6. *In any variant of* BiBBI *obeying both weak distribution and unit contraction, we have the following multiplicative analogue of the usual* disjunctive syllogism*:*

$$A * (\sim\! A \overset{*}{\vee} B) \vdash B$$

Proof. First, $A \mathrel{-\!\!*} \bot^* \vdash A \mathrel{-\!\!*} \bot^*$ is trivially provable. Thus, by residuation and commutativity of $*$, we obtain $A * (A \mathrel{-\!\!*} \bot^*) \vdash \bot^*$, which is equal to $A * \sim\! A \vdash \bot^*$. Since $B \vdash B$ is trivially provable, we obtain $(A * \sim\! A) \overset{*}{\vee} B \vdash \bot^* \overset{*}{\vee} B$ using BiBBI's monotonicity rule for $\overset{*}{\vee}$. Now, as $\bot^* \overset{*}{\vee} B \vdash B$ is an instance of the unit contraction axiom and $A * (\sim\! A \overset{*}{\vee} B) \vdash (A * \sim\! A) \overset{*}{\vee} B$ is an instance of the weak distribution axiom, we obtain $A * (\sim\! A \overset{*}{\vee} B) \vdash B$ by transitivity. □

Principle	Axiom $\mathcal{A}$	Frame condition $\mathcal{F}(\mathcal{A})$
Associativity	$A \mathbin{\overset{*}{\vee}} (B \mathbin{\overset{*}{\vee}} C) \vdash (A \mathbin{\overset{*}{\vee}} B) \mathbin{\overset{*}{\vee}} C$	$w_1 \triangledown (w_2 \triangledown w_3) = (w_1 \triangledown w_2) \triangledown w_3$
Unit expansion	$A \vdash A \mathbin{\overset{*}{\vee}} \bot^*$	$w \triangledown U \subseteq \{w\}$
Unit contraction	$A \mathbin{\overset{*}{\vee}} \bot^* \vdash A$	$w \in w \triangledown U$
Contraction	$A \mathbin{\overset{*}{\vee}} A \vdash A$	$w \in w \triangledown w$
Weak distribution	$A * (B \mathbin{\overset{*}{\vee}} C) \vdash (A * B) \mathbin{\overset{*}{\vee}} C$	$(x_1 \circ x_2) \cap (y_1 \triangledown y_2) \neq \emptyset$ implies $\exists w.\ y_1 \in x_1 \circ w$ and $x_2 \in w \triangledown y_2$
Classicality	$\sim\sim A \vdash A$	$\exists! -w.\ (w \circ -w) \cap U \neq \emptyset$

Table 1: Optional axioms of BiBBI and the corresponding first-order frame conditions (from which we suppress outermost universal quantifiers over the model domain).

The interpretations of $\bot^*$ and $\mathbin{\overset{*}{\vee}}$ in BiBBI are (partially) justified by the following proposition, which shows that BiBBI with classicality is exactly CBI.

Proposition 3.7. *The following relationships hold between* BiBBI *and* CBI*:*

1. *For any* BiBBI*-model $\langle W, \circ, E, \triangledown, U\rangle$ satisfying classicality in Table 1, $\langle W, \circ, E, U\rangle$ is a* CBI*-model.*

2. *Any* CBI*-model $\langle W, \circ, E, U\rangle$ can be viewed as a* BiBBI*-model $\langle W, \circ, E, \triangledown, U\rangle$, by taking $w_1 \triangledown w_2 = -(-w_1 \circ -w_2)$. This model satisfies the frame conditions for classicality, associativity, unit expansion / contraction, and weak distribution.*

3. *When* CBI*-models are identified with* BiBBI*-models as above,* CBI*-validity (Defn. 2.10) coincides with validity in the corresponding variant of* BiBBI.

4. *Any* CBI*-formula that is valid in a variant of* BiBBI *without contraction is also* CBI*-valid.*

Proof. 1. Immediate by construction.

2. Let $\langle W, \circ, E, U\rangle$ be a CBI-model. It is immediate that $\langle W, \circ, E, \triangledown, U\rangle$ is a basic BiBBI-model, with commutativity of $\triangledown$ an easy consequence of the commutativity of $\circ$. We have to check that $\langle W, \circ, E, \triangledown, U\rangle$ is indeed a model of BiBBI satisfying the required properties.

   Next, we have to check that $\langle W, \circ, E, \triangledown, U\rangle$ satisfies all of the frame conditions mentioned above. Classicality is exactly the CBI-model axiom, so is trivially satisfied (and consequently we have $--w = \{w\}$ for any $w \in W$ and $-E = U$, cf. [4]). For associativity, we check as follows:

$$
\begin{aligned}
w_1 \triangledown (w_2 \triangledown w_3) &= -(-w_1 \circ --(-w_2 \circ -w_3)) \\
&= -(-w_1 \circ (-w_2 \circ -w_3)) && \text{(since } --X = X\text{)} \\
&= -((-w_1 \circ -w_2) \circ -w_3) && \text{(by associativity of } \circ\text{)} \\
&= -(--(-w_1 \circ -w_2) \circ -w_3) && \text{(since } --X = X\text{)} \\
&= (w_1 \triangledown w_2) \triangledown w_3
\end{aligned}
$$

Next, we check the unit contraction and unit expansion axioms simultaneously:

$$\begin{aligned}
U \mathbin{\triangledown} w &= \textstyle\bigcup_{u \in U} -(-u \circ -w) \\
&= -(\textstyle\bigcup_{u \in U} (-u \circ -w)) \\
&= -((\textstyle\bigcup_{u \in U} -u) \circ -w) \\
&= -(E \circ -w) && (\text{since } \textstyle\bigcup_{u \in U} -u = -U = E) \\
&= \{--w\} = \{w\}
\end{aligned}$$

Finally, we must verify the weak distribution axiom. Suppose $(x_1 \circ x_2) \cap (y_1 \mathbin{\triangledown} y_2) \neq \emptyset$. That is, for some $z \in x_1 \circ x_2$ we have $z \in -(-y_1 \circ -y_2)$, or equivalently $-z \in -y_1 \circ -y_2$, which is again equivalent (see [4]) to $y_1 \in z \circ -y_2$. Putting everything together and using associativity of $\circ$, we get $y_1 \in x_1 \circ (x_2 \circ -y_2)$. Thus, for some $w \in x_2 \circ -y_2$, we have $y_1 \in x_1 \circ w$. But, using the same properties as before, $w \in x_2 \circ -y_2$ is equivalent to $-x_2 \in -w \circ -y_2$ and then to $x_2 \in -(-w \circ -y_2)$, i.e. $x_2 \in w \mathbin{\triangledown} y_2$ as required. This completes the verification of the frame conditions.

3. Just observe that the clauses for satisfaction of $\bot^*$ coincide in the forcing relations for BiBBI and CBI, and that by inserting the definition of $\triangledown$ into BiBBI's clause for $\mathbin{\nabla\!\!\!*}$, we obtain exactly the usual CBI clause for $\mathbin{\nabla\!\!\!*}$.

4. Let A be a CBI-formula valid in a contraction-free variant of BiBBI. Let M be a CBI-model. Using part 2 of the proposition, we can extend M to a BiBBI-model M' satisfying all properties in Table 1 except contraction; therefore, M' is a model of the required BiBBI variant. By assumption, A is valid in M' (w.r.t. the contraction-free BiBBI variant). By part 3 of the proposition, A is then valid in the CBI-model M, as required. □

Theorem 3.8. *For each principle listed in Table 1, the logical axiom $\mathcal{A}$ corresponding to that principle* defines *the corresponding frame condition $\mathcal{F}(\mathcal{A})$. That is, for any basic* BiBBI-*model M, the axiom $\mathcal{A}$ is valid in M if and only if M has the property $\mathcal{F}(\mathcal{A})$.*

Proof. Let $M = \langle W, \circ, E, \triangledown, U \rangle$ be a basic BiBBI-model. We distinguish a case for each principle from Table 1.

Associativity: ($\Leftarrow$) Assuming $\triangledown$ is associative, we have to show $A \mathbin{\nabla\!\!\!*} (B \mathbin{\nabla\!\!\!*} C) \vdash (A \mathbin{\nabla\!\!\!*} B) \mathbin{\nabla\!\!\!*} C$ is valid in M. So, assuming $w \models_\rho A \mathbin{\nabla\!\!\!*} (B \mathbin{\nabla\!\!\!*} C)$, we have to show that $w \models_\rho (A \mathbin{\nabla\!\!\!*} B) \mathbin{\nabla\!\!\!*} C$. This means, assuming $w \in u \mathbin{\triangledown} v$, we have to show that $u \models_\rho A \mathbin{\nabla\!\!\!*} B$ or $v \models_\rho C$. If $v \models_\rho C$ we are done. Otherwise we have $v \not\models_\rho C$ and must show $u \models_\rho A \mathbin{\nabla\!\!\!*} B$. Thus, assuming $u \in a \mathbin{\triangledown} b$, we have to show $a \models_\rho A$ or $b \models_\rho B$. Now, collecting assumptions and using associativity of $\triangledown$, we have:

$$w \in (a \mathbin{\triangledown} b) \mathbin{\triangledown} v = a \mathbin{\triangledown} (b \mathbin{\triangledown} v)$$

That is, we have $w \in a \mathbin{\triangledown} x$ for some $x \in b \mathbin{\triangledown} v$. Since $w \models_\rho A \mathbin{\nabla\!\!\!*} (B \mathbin{\nabla\!\!\!*} C)$ by assumption and $w \in a \mathbin{\triangledown} x$, we have either $a \models_\rho A$ or $x \models_\rho B \mathbin{\nabla\!\!\!*} C$. If $a \models_\rho A$ we are done. Otherwise, since $x \models_\rho B \mathbin{\nabla\!\!\!*} C$ and $x \in b \mathbin{\triangledown} v$ and $v \not\models_\rho C$, we have $b \models_\rho B$ as required.

($\Rightarrow$) Assuming that $A \mathbin{\nabla\!\!\!*} (B \mathbin{\nabla\!\!\!*} C) \vdash (A \mathbin{\nabla\!\!\!*} B) \mathbin{\nabla\!\!\!*} C$ is valid in M, we must show that $\triangledown$ is associative. Since $\triangledown$ is commutative by definition, it suffices to show for arbitrary $w, x, y \in W$ that $(w \mathbin{\triangledown} x) \mathbin{\triangledown} y \subseteq w \mathbin{\triangledown} (x \mathbin{\triangledown} y)$. Let $z \in (w \mathbin{\triangledown} x) \mathbin{\triangledown} y$, so that there exists $u \in W$ with $z \in u \mathbin{\triangledown} y$ and $u \in w \mathbin{\triangledown} x$. Let A, B, C be propositional variables and define a valuation ρ for M by:

$$\rho(A) = W \setminus \{w\} \qquad \rho(B) = W \setminus \{x\} \qquad \rho(C) = W \setminus \{y\}$$

Now, since $u \in w \triangledown x$ but $w \not\models_\rho A$ and $x \not\models_\rho B$, we have $u \not\models_\rho A \overset{*}{\vee} B$. Similarly, since $z \in u \triangledown y$ but $u \not\models_\rho A \overset{*}{\vee} B$ and $y \not\models_\rho C$ we have $z \not\models_\rho (A \overset{*}{\vee} B) \overset{*}{\vee} C$. Since the associativity axioms $A \overset{*}{\vee} (B \overset{*}{\vee} C) \vdash (A \overset{*}{\vee} B) \overset{*}{\vee} C$ is valid in M, we must have $z \not\models_\rho A \overset{*}{\vee} (B \overset{*}{\vee} C)$. This implies there exist $w_1, w_2, w', w'' \in W$ with

$$z \in w_1 \triangledown w_2 \text{ and } w_1 \not\models_\rho A \text{ and } w_2 \in w' \triangledown w'' \text{ and } w' \not\models_\rho B \text{ and } w'' \not\models_\rho C$$

That is, $z \in w \triangledown w_2$ and $w_2 \in x \triangledown y$ for some $w_2 \in W$, which means $z \in w \triangledown (x \triangledown y)$. This completes the case.

Unit expansion: ($\Leftarrow$) Assuming that $w \triangledown U \subseteq \{w\}$ for all $w \in W$, we have to show that $A \vdash A \overset{*}{\vee} \bot^*$ is valid in M. So, assuming that $w \models_\rho A$, we have to show $w \models_\rho A \overset{*}{\vee} \bot^*$. This means showing, assuming that $w \in w_1 \triangledown w_2$, that either $w_1 \models_\rho A$ or $w_2 \models_\rho \bot^*$. If $w_2 \models_\rho \bot^*$ then we are done, so assume that $w_2 \not\models_\rho \bot^*$, which means $w_2 \in U$. In that case, $w \in w_1 \triangledown U \subseteq \{w_1\}$, which implies $w_1 = w$ and thus $w_1 \models_\rho A$ as required.

($\Rightarrow$) Assuming that $A \vdash A \overset{*}{\vee} \bot^*$ is valid in M, we have to show that $w \triangledown U \subseteq \{w\}$ for any $w \in W$. Let $x \in w \triangledown U$, which means $x \in w \triangledown u$ for some $u \in U$. We have to show that $x = w$. Let A be a propositional variable and define a valuation ρ for M by $\rho(A) = W \setminus \{w\}$. Since $x \in w \triangledown u$ but $w \not\models_\rho A$ and $u \not\models_\rho \bot^*$, we have $x \not\models_\rho A \overset{*}{\vee} \bot^*$. Since $A \vdash A \overset{*}{\vee} \bot^*$ is valid in M, we have $x \not\models_\rho A$, i.e. $x = w$ as required.

Unit contraction: ($\Leftarrow$) Assuming $w \in w \triangledown U$ for all $w \in W$, we have to show that $A \overset{*}{\vee} \bot^* \vdash A$ is valid in M. So, assuming that $w \models_\rho A \overset{*}{\vee} \bot^*$, we must show $w \models_\rho A$. As $w \in w \triangledown u$ for some $u \in U$ and $w \models_\rho A \overset{*}{\vee} \bot^*$, we must have either $w \models_\rho A$ or $u \models_\rho \bot^*$. The latter is impossible, so $w \models_\rho A$ as required.

($\Rightarrow$) Assuming $A \overset{*}{\vee} \bot^* \vdash A$ is valid in M, we have to show that $w \in w \triangledown U$ for any $w \in W$. Let A be a propositional variable and define a valuation ρ for M by $\rho(A) = W \setminus \{w\}$. By construction, $w \not\models_\rho A$. Since $A \overset{*}{\vee} \bot^* \vdash A$ is valid in M, we have $w \not\models_\rho A \overset{*}{\vee} \bot^*$. This means there exist $x, u \in W$ such that $w \in x \triangledown u$ and $x \not\models_\rho A$ and $u \not\models_\rho \bot^*$. That is, $w \in w \triangledown u$ for some $u \in U$, i.e. $w \in w \triangledown U$ as required.

Contraction: ($\Leftarrow$) Assuming $w \in w \triangledown w$ for all $w \in W$, we have to show that $A \overset{*}{\vee} A \vdash A$ is valid in M. Assuming that $w \models_\rho A \overset{*}{\vee} A$, since $w \in w \triangledown w$ we immediately have $w \models_\rho A$ as required.

($\Rightarrow$) Assuming that $A \overset{*}{\vee} A \vdash A$ is valid in M, we have to show that $w \in w \triangledown w$ for any $w \in W$. Let A be a propositional variable and define a valuation ρ for M by $\rho(A) = W \setminus \{w\}$. By construction, $w \not\models_\rho A$. Since $A \overset{*}{\vee} A \vdash A$ is valid in M, we have $w \not\models_\rho A \overset{*}{\vee} A$. This means there exist $w_1, w_2 \in W$ such that $w \in w_1 \triangledown w_2$ and $w_1 \not\models_\rho A$ and $w_2 \not\models_\rho A$. That is, $w = w_1 = w_2$, and so $w \in w \triangledown w$ as required.

Weak distribution: ($\Leftarrow$) Assuming that the weak distribution frame property holds in M, we have to show that $A * (B \overset{*}{\vee} C) \vdash (A * B) \overset{*}{\vee} C$ is valid in M. So, given $w \models_\rho A * (B \overset{*}{\vee} C)$, we must show that $w \models_\rho (A * B) \overset{*}{\vee} C$. This means showing, assuming that $w \in w_1 \triangledown w_2$, that either $w_1 \models_\rho A * B$ or $w_2 \models_\rho C$. Since $w \models_\rho A * (B \overset{*}{\vee} C)$, we have $w \in x_1 \circ x_2$ where $x_1 \models_\rho A$ and $x_2 \models_\rho B \overset{*}{\vee} C$. Collecting assumptions, we have $(x_1 \circ x_2) \cap (w_1 \triangledown w_2) \neq \emptyset$, so by the weak distribution frame property there exists $y \in W$ such that $w_1 \in x_1 \circ y$ and $x_2 \in y \triangledown w_2$. Now, since $x_2 \in y \triangledown w_2$ and $x_2 \models_\rho B \overset{*}{\vee} C$ we must have either $y \models_\rho B$

or $w_2 \models_\rho C$. If $w_2 \models_\rho C$, we are done. Otherwise, since $w_1 \in x_1 \circ y$ and $x_1 \models_\rho A$ and $y \models_\rho B$, we have $w_1 \models_\rho A * B$ as required.

($\Rightarrow$) Assuming that $A * (B \mathbin{\overset{*}{\vee}} C) \vdash (A * B) \mathbin{\overset{*}{\vee}} C$ is valid in M, we have to show that the weak distribution frame property holds in M. That is, supposing $z \in (x_1 \circ x_2) \cap (y_1 \triangledown y_2)$, we have to find a $w \in W$ such that $y_1 \in x_1 \circ w$ and $x_2 \in w \triangledown y_2$. Let A, B, C be propositional variables and define a valuation ρ for M by

$$\rho(A) = \{x_1\} \qquad \rho(B) = \{w \in W \mid x_2 \in w \triangledown y_2\} \qquad \rho(C) = W \setminus \{y_2\}$$

Now observe that by construction of ρ we have the following:

$$\begin{array}{ll} & \forall w_1 \in W.\ x_2 \in w_1 \triangledown y_2 \text{ implies } w_1 \models_\rho B \\ \text{i.e.} & \forall w_1, w_2 \in W.\ x_2 \in w_1 \triangledown w_2 \text{ and } w_2 \not\models_\rho C \text{ implies } w_1 \models_\rho B \\ \text{i.e.} & \forall w_1, w_2 \in W.\ x_2 \in w_1 \triangledown w_2 \text{ implies } w_1 \models_\rho B \text{ or } w_2 \models_\rho C \\ \text{i.e.} & x_2 \models_\rho B \mathbin{\overset{*}{\vee}} C \end{array}$$

Since $z \in x_1 \circ x_2$, with $x_1 \models_\rho A$ by construction and $x_2 \models_\rho B \mathbin{\overset{*}{\vee}} C$ by the above, we get $z \models_\rho A * (B \mathbin{\overset{*}{\vee}} C)$. By the main assumption, $z \models_\rho (A * B) \mathbin{\overset{*}{\vee}} C$. Now, as $z \models_\rho (A * B) \mathbin{\overset{*}{\vee}} C$ and $z \in y_1 \triangledown y_2$ but $y_2 \not\models_\rho C$, we must have $y_1 \models_\rho A * B$. This means that there exist $u, w \in W$ with $y_1 \in u \circ w$ and $u \models_\rho A$ and $w \models_\rho B$. By definition of ρ, this precisely means that $y_1 \in x_1 \circ w$ and $x_2 \in w \triangledown y_2$, as required.

Classicality: ($\Leftarrow$) Assuming the CBI-model axiom holds in M, we have to show that $\sim\sim A \vdash A$ is valid. This means, assuming that $w \models_\rho \sim\sim A$, showing that $w \models_\rho A$. Using the clause for satisfaction of $\sim$ given in Section 2, we have $--w \models_\rho A$, and thus immediately $w \models_\rho A$ using the fact from [4] that $-$ is an involutive function on W.

($\Rightarrow$) Assuming that $\sim\sim A \vdash A$ is valid in M, we have to show that the CBI-model axiom holds, i.e. that for any $w \in W$ there is a unique $w' \in W$ such that $(w \circ w') \cap U \neq \emptyset$. Let A be a propositional variable and define a valuation ρ for M by $\rho(A) = W \setminus \{w\}$. By construction, $w \not\models_\rho A$, so using the main assumption we have $w \not\models_\rho (A \mathrel{-\!\!*} \bot^*) \mathrel{-\!\!*} \bot^*$. Thus, there exist $w', w'' \in W$ such that $w'' \in w \circ w'$ and $w' \models_\rho A \mathrel{-\!\!*} \bot^*$ but $w'' \not\models_\rho \bot^*$, i.e. $w'' \in U$. That is, there exists an $-w = w' \in W$ such that $(w \circ -w) \cap U \neq \emptyset$.

It just remains to show that $-w$ is unique. Write $\mathrm{Co}(w)$ for the set of all w' such that $(w \circ w') \cap U \neq \emptyset$, and note that $\mathrm{Co}(w)$ is nonempty by the above. According to part 1 of Proposition 2.2 in [4], it suffices to show that $\mathrm{Co}(\mathrm{Co}(w)) \subseteq \{w\}$, extending Co pointwise to sets as usual. To see this, first define a new valuation ρ' for M by $\rho'(A) = \{w\}$, so that $w \models_{\rho'} A$ by construction. Since $A \vdash \sim\sim A$ is already provable in BBI, we have $w \models_{\rho'} (A \mathrel{-\!\!*} \bot^*) \mathrel{-\!\!*} \bot^*$. It is easy to show that this means that $w' \models_{\rho'} A$ for all $w' \in \mathrm{Co}(\mathrm{Co}(w))$. That is, $\mathrm{Co}(\mathrm{Co}(w)) \subseteq \{w\}$ as required. □

Corollary 3.9 (Soundness of BiBBI)**.** *If a formula is provable in some variant of* BiBBI *then it is valid in that variant.*

Proof. Follows immediately from Theorems 3.4 and 3.8. □

We conclude this section by stating our completeness result, the converse to Corollary 3.9.

Theorem 3.10 (Completeness of BiBBI)**.** *If a* BiBBI*-formula is valid in some variant of* BiBBI *then it is provable in that variant.*

We present the detailed proof of Theorem 3.10 in Section 5.

4 From BBI-models to BiBBI-models

In this section, we consider whether interesting models of BiBBI can be obtained via general constructions on BBI-models.

First, we observe that any pair of BBI-models $M = \langle W, \circ, E\rangle$ and $M' = \langle W, \circ', E'\rangle$ over the same set of worlds can be trivially "joined" into a BiBBI-model $\langle W, \circ, E, \circ', E'\rangle$. Of the logical principles in Table 1, this BiBBI-model satisfies associativity, unit contraction and unit expansion, but no other properties in general; in particular, *weak distribution* typically fails, and so we cannot reason very meaningfully about any interaction between M and M'.

Below, we show how to turn any *partial functional* BBI model $M = \langle W, \circ, E\rangle$ into a BiBBI-model $\langle W', \circ', E', \triangledown, U\rangle$ that does obey weak distribution, using two general constructions. The first construction, given in Section 4.1, extends M directly to a BiBBI-model $\langle W, \circ, E, \triangledown, U\rangle$ in which $\triangledown$ is interpreted as a type of generalised "intersection". This construction yields BiBBI-models with the contraction and weak distribution properties, but in general no others (Theorem 4.5).

In the second construction (Section 4.2), which builds on the first one, the worlds W' of the constructed model are *pairs* of worlds from the original BBI-model, where the first component of each pair is "included" in the second component. Then, $\triangledown$ is interpreted as intersection (as defined in the first construction) on the first component and the identity on the second one. Moreover, we show that if the original BBI-model satisfies the *cross-split* and *disjointness* properties typical of heap-like models (see Definition 4.6), then the constructed model satisfies *all* the principles listed in Table 1, except for classicality (Theorem 4.12).

For both constructions, we give examples based on the heap model from Example 2.4.

4.1 Intersection in BBI-models

Our first approach to constructing BiBBI-models from BBI-models is to interpret $\triangledown$ as an "intersection-like" operator on worlds. As a motivating example, there are two natural ways one could go about defining an such an operator in the heap model of Example 2.4, depending on how to deal with *incompatible heaps*:

Example 4.1 (Intersections of heaps)**.** Two heaps h_1, h_2 are said to be *compatible at* ℓ if $h_1(\ell) = h_2(\ell)$, and simply *compatible* if they are compatible for all $\ell \in \mathsf{dom}(h_1) \cap \mathsf{dom}(h_2)$. In particular, any two heaps with disjoint domains are compatible.

We define two intersection operations $\cap_1$ and $\cap_2$ on heaps as follows:

$$(h_1 \cap_1 h_2)(\ell) =_{\text{def}} \begin{cases} h_1(\ell) & \text{if } \ell \in \mathsf{dom}(h_1) \cap \mathsf{dom}(h_2) \text{ and } h_1, h_2 \text{ are compatible at } \ell \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$(h_1 \cap_2 h_2)(\ell) =_{\text{def}} \begin{cases} (h_1 \cap_1 h_2)(\ell) & \text{if } h_1 \text{ and } h_2 \text{ are compatible} \\ \text{undefined} & \text{otherwise} \end{cases}$$

compatible, and $\emptyset$ otherwise. Both interpretations of heap intersection make intuitive sense: in the first case, incompatibilities are silently discarded, while the second intersection detects them and aborts the whole composition. However, $\cap_1$ is associative, while $\cap_2$ is not. We note, however, that neither $\cap_1$ nor $\cap_2$ has a natural set of units $U \subseteq \text{Heaps}$, in the sense that $h \cap_i U = \{h\}$ for all heaps h.

Proposition 4.2. *Let* $\langle \text{Heaps}, \circ, \{e\}\rangle$ *be the heap model of Example 2.4, and let* $\cap_1$ *and* $\cap_2$ *be the heap intersection operations defined in Example 4.1. Then, (for any set* $U \subseteq \text{Heaps}$*) both* $\langle \text{Heaps}, \circ, \{e\}, \cap_1, U\rangle$ *and* $\langle \text{Heaps}, \circ, \{e\}, \cap_2, U\rangle$ *are* BiBBI*-models with the contraction and weak distribution properties.*

Proof. Straightforward verifications. □

We can extend the above proposition to the case of arbitrary partial functional BBI-models, using a generalised version of the heap intersection $\cap_2$.

Definition 4.3. Let $M = \langle W, \circ, E\rangle$ be a BBI-model. For any $w_1, w_2 \in W$, define $w_1 \triangledown_\cap w_2$ as

$$\{x \mid \exists x_1, x_2 \in W.\ w_1 \in x \circ x_1 \text{ and } w_2 \in x \circ x_2 \text{ and } w_1 \circ x_2 \neq \emptyset \text{ and } w_2 \circ x_1 \neq \emptyset\}\ .$$

Example 4.4. In the heap model of Example 2.4, $h_1 \triangledown_\cap h_2$ is exactly $h_1 \cap_2 h_2$.

From now on, to simplify notations, and because most models of separation logic in the literature satisfy this constraint, we will assume an underlying BBI-model that is *partial functional*. Thus we write, e.g., $w_1 \circ w_2 = w$ rather than $w_1 \circ w_2 = \{w\}$. Taking advantage of associativity of $\circ$, we also write $w_1 \sharp \ldots \sharp w_n$ to mean that $w_1 \circ \ldots \circ w_n$ is defined (i.e., non-empty). The operation $\triangledown_\cap$ in Definition 4.3 can then be rewritten as follows:

$$w_1 \triangledown_\cap w_2 \ =_{\text{def}}\ \{x \mid \exists x_1, x_2 \in W.\ w_1 = x \circ x_1 \text{ and } w_2 = x \circ x_2 \text{ and } x \sharp x_1 \sharp x_2\}$$

(Note that $\triangledown_\cap$ is itself not a partial function in general, and nor is it necessarily associative.)

Theorem 4.5. *For any partial functional* BBI-*model* $M = \langle W, \circ, E\rangle$*, and any* $U \subseteq W$*, we have that* $\langle W, \circ, E, \triangledown, U\rangle$ *is a* BiBBI-*model with the contraction and weak distribution properties.*

Proof. Since M is a BBI-model and $\triangledown_\cap \colon W \times W \to \mathcal{P}(W)$ is commutative by construction, $\langle W, \circ, E, \triangledown, U\rangle$ is a basic BiBBI-model.

To check that contraction holds, we need to show that $w \in w \triangledown_\cap w$ for any $w \in W$. Since M is a BBI-model, there is an $e_w \in E$ such that $w \circ e_w = w$. Then,

$$w = w \circ e_w \text{ and } w = w \circ e_w \text{ and } w \sharp e_w \sharp e_w$$

hence $w \in w \triangledown_\cap w$ as required.

It just remains to verify the weak distribution law. That is, assuming $(x_1 \circ x_2) \cap (y_1 \triangledown_\cap y_2) \neq \emptyset$, we require to find $w \in W$ such that $y_1 = x_1 \circ w$ and $x_2 \in w \triangledown_\cap y_2$. By assumption, we have $(x_1 \circ x_2) = z \in y_1 \triangledown_\cap y_2$ (for some z). By definition of $\triangledown_\cap$ there are z_1 and z_2 such that

$$y_1 = z \circ z_1 \text{ and } y_2 = z \circ z_2 \text{ and } z \sharp z_1 \sharp z_2\ .$$

Now we let $w = x_2 \circ z_1$. We immediately have

$$y_1 = z \circ z_1 = x_1 \circ x_2 \circ z_1 = x_1 \circ w\ .$$

To see that $x_2 \in w \triangledown_\cap y_2$, we need to find $x', x'' \in W$ such that

$$w = x_2 \circ x' \text{ and } y_2 = x_2 \circ x'' \text{ and } x_2 \sharp x' \sharp x''\ .$$

We choose $x' = z_1$ and $x'' = x_1 \circ z_2$. We have $w = x_2 \circ z_1$ as required by construction. We also have

$$y_2 = z \circ z_2 = x_1 \circ x_2 \circ z_2 = x_2 \circ (x_1 \circ z_2)\ .$$

It remains to check $x_2 \sharp z_1 \sharp (x_1 \circ z_2)$, or equivalently $(x_1 \circ x_2) \sharp z_1 \sharp z_2$, which follows from $x_1 \circ x_2 = z$ and $z \sharp z_1 \sharp z_2$. □

4.2 Intersection in BBI-models with global worlds

We are now ready to define our second general construction, based upon the one in the previous section, for constructing BiBBI-models obeying weak distribution, associativity, contraction *and* the unit laws (see Table 1). We begin with a partial functional BBI-model $M = \langle W, \circ, E\rangle$ obeying the *cross-split* and *disjointness* properties typically encountered in heap-like models of separation logic [10, 6], and construct a BiBBI-model $\bar{M} = \langle \bar{W}, \bar{\circ}, \bar{E}, \bar{\triangledown}, D\rangle$. Each world in $\bar{W}$ consists of a "local" world $w \in W$ paired with a "global" world $x \in W$ that "extends" w in the sense that $x = w \circ w'$ for some w'. On the "local" part of each world, $\bar{\circ}$ and $\bar{\triangledown}$ behave as $\circ$ and $\triangledown_{\cap}$, respectively. On the "global" part of each world, $\bar{\circ}$ and $\bar{\triangledown}$ behave as a *union* operation $\cup$ (as defined below) and the identity, respectively.

First, we recall the cross-split and disjointness properties from [10]. These properties are typical of heap-like models (which have considerably more structure than general BBI-models), and are needed in order to make our general construction work as intended.

Definition 4.6 ([10])**.** A partial functional BBI-model $\langle W, \circ, E\rangle$ has the *cross-split property* if for any $t, u, v, w \in W$ such that $t \circ u = v \circ w$, there exist tv, tw, uv, uw such that

$$t = tv \circ tw, \quad u = uv \circ uw, \quad v = tv \circ uv, \quad \text{and} \quad w = tw \circ uw.$$

M has the *disjointness property* (a.k.a. positivity [10]) if $w \circ w$ is undefined for all $w \notin E$.

Next, we define a generalised notion of "union" for BBI-models.

Definition 4.7. Given a partial functional BBI-model $\langle W, \circ, E\rangle$, we define the *union* operation, $\cup : W \times W \to \mathcal{P}(W)$, by

$$w_1 \cup w_2 =_{\text{def}} \{y \circ y_1 \circ y_2 \mid w_1 = y \circ y_1 \text{ and } w_2 = y \circ y_2\} \ .$$

We lift $\cup$ to $\mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(W)$ in the same way as for $\circ$:

$$W_1 \cup W_2 =_{\text{def}} \bigcup_{w_1 \in W_1, w_2 \in W_2} w_1 \cup w_2 \ .$$

For our purposes we shall require $\cup$ to be associative, which is not necessarily the case for arbitrary partial functional BBI-models. However, as we are about to see, whenever the underlying BBI-model has the cross-split property, then $\cup$ becomes associative (an observation made previously in [14]).

Lemma 4.8. *If a partial functional* BBI-*model* $M = \langle W, \circ, E\rangle$ *has the cross-split property, then* $\cup$ *in Definition 4.7 is associative. Moreover, if* $w = w_1 \circ w_2$*, then* $w \in w \cup w_1$*.*

Proof. First, we show associativity of $\cup$. As $\cup$ is easily seen to be commutative, it suffices to show $w_1 \cup (w_2 \cup w_3) \subseteq (w_1 \cup w_2) \cup w_3$.

Suppose $w \in w_1 \cup (w_2 \cup w_3)$, that is, $w \in w_1 \cup w'$ for some $w' \in w_2 \cup w_3$. By the first of these, we have $w = y \circ y_1 \circ y_1'$, where $w_1 = y \circ y_1$ and $w' = y \circ y_1'$. By the second, we obtain $w' = y' \circ y_2 \circ y_3$, where $w_2 = y' \circ y_2$ and $w_3 = y' \circ y_3$. Thus we obtain

$$w' = y \circ y_1' = y' \circ (y_2 \circ y_3)$$

Thus by the cross-split property, we obtain $a, b, c, d \in W$ with

$$y = a \circ b, \quad y' = a \circ c, \quad y_1' = c \circ d, \quad \text{and} \quad y_2 \circ y_3 = b \circ d \ .$$

By applying cross-split to the last of these equalities, we get $\alpha, \beta, \gamma, \delta \in W$ such that

$$b = \alpha \circ \beta, \quad d = \gamma \circ \delta, \quad y_2 = \alpha \circ \gamma, \quad \text{and} \quad y_3 = \beta \circ \delta \ .$$

Now, collecting equalities and using associativity / commutativity of $\circ$, we have the following:

$$\begin{aligned}w_1 &= y \circ y_1 = a \circ b \circ y_1 = a \circ \alpha \circ \beta \circ y_1 = (a \circ \alpha) \circ (y_1 \circ \beta)\\ w_2 &= y' \circ y_2 = a \circ c \circ \alpha \circ \gamma = (a \circ \alpha) \circ (c \circ \gamma)\end{aligned}$$

Thus, by definition of $\cup$, we obtain

$$(a \circ \alpha) \circ (y_1 \circ \beta) \circ (c \circ \gamma) \in w_1 \cup w_2$$

Now write w'' for $(a \circ \alpha) \circ (y_1 \circ \beta) \circ (c \circ \gamma)$, and note that we have the following:

$$\begin{aligned}w'' &= (a \circ c \circ \beta) \circ (y_1 \circ \alpha \circ \gamma)\\ w_3 &= y' \circ y_3 = a \circ c \circ \beta \circ \delta = (a \circ c \circ \beta) \circ \delta\\ w &= y \circ y_1 \circ y_1' = a \circ b \circ y_1 \circ c \circ d = a \circ \alpha \circ \beta \circ y_1 \circ c \circ \gamma \circ \delta\\ &= (a \circ c \circ \beta) \circ (\alpha \circ y_1 \circ \gamma) \circ \delta\end{aligned}$$

Thus we have $w \in w'' \cup w_3$. Since $w'' \in w_1 \cup w_2$ by the above, we have $w \in (w_1 \cup w_2) \cup w_3$ as required.

For the second part of the lemma, suppose $w = w_1 \circ w_2$. There is an $e \in E$ such that $w_1 = w_1 \circ e$. Then, by construction, $w_1 \circ w_2 \circ e \in w \cup w_1$. The result follows by observing $w_1 \circ w_2 \circ e = (w_1 \circ e) \circ w_2 = w_1 \circ w_2 = w$. □

Definition 4.9. Let $M = \langle W, \circ, E\rangle$ be a partial functional BBI-model. We define $\bar{M} = \langle \bar{W}, \bar{\circ}, \bar{E}, \bar{\triangledown}, D\rangle$ as follows:

$$\begin{aligned}\bar{W} &=_{\text{def}} \{(w,x) \mid \exists w'.\ x = w \circ w'\}\\ (w,x)\,\bar{\circ}\,(w',x') &=_{\text{def}} \{(w \circ w', x'') \mid x'' \in x \cup x'\}\\ \bar{E} &=_{\text{def}} \{(e,e) \mid e \in E\}\\ (w,x)\,\bar{\triangledown}\,(w',x') &=_{\text{def}} \begin{cases}\{(w'',x) \mid w'' \in w \triangledown_{\cap} w'\} & \text{if } x = x'\\ \emptyset & \text{otherwise}\end{cases}\\ D &=_{\text{def}} \{(w,w) \mid w \in W\}\end{aligned}$$

Theorem 4.10. *Given a partial functional* BBI*-model M with the cross-split and disjointness properties, $\bar{M}$ is a* BiBBI*-model with the unit contraction, contraction, and weak distribution properties.*

Proof. Let $M = \langle W, \circ, E\rangle$. First we have to check that $\bar{M}$ is a basic BiBBI-model. It is clear that both $\bar{\circ}$ and $\bar{\triangledown}$ are commutative. Associativity of $\bar{\circ}$ follows from the associativity of $\circ$ and of $\cup$ (Lemma 4.8). We just need to check that $(w,x)\,\bar{\circ}\,\bar{E} = \{(w,x)\}$ for any $(w,x) \in \bar{W}$.

Let $(e,e) \in \bar{E}$, and note that $w \circ e$ is either undefined or w. If $w \circ e$ is undefined then so is $(w,x)\,\bar{\circ}\,(e,e)$, by construction. There is at least one $(e,e) \in \bar{E}$ such that $w \circ e = w$, and in that case we have $(w,x)\,\bar{\circ}\,(e,e) = \{(w,x') \mid x' \in x \cup e\}$. It thus suffices to show that $x \cup e = \{x\}$. Note that $x = w \circ w'$ for some $w' \in W$. We have by definition

$$x \cup e = \{y \circ y_1 \circ y_2 \mid x = y \circ y_1 \text{ and } e = y \circ y_2\}.$$

Since $e = e \circ e$ and $x = x \circ e$ (because $x = w \circ w'$), we have $x \circ e \circ e = x \in x \cup e$. Now, supposing $z \in x \cup e$, we must show $z = x$. By construction we have $z = y \circ y_1 \circ y_2$ where $x = y \circ y_1$ and $e = y \circ y_2$. Note that $(y \circ y_2) \circ (y \circ y_2)$ is defined (and equal to e), which means that $y \circ y$ and $y_2 \circ y_2$ are also both defined. By disjointness of M, we must have $y, y_2 \in E$, which implies $y = y_2 = e$. Thus $x = e \circ y_1 = y_1$, and so $z = e \circ y_1 \circ e = x$, as required.

We now establish the unit contraction, contraction and weak distribution properties for $\bar{M}$.

Unit contraction: Given $(w,x) \in \bar{W}$, we have to show that $(w,x) \in (w,x) \mathbin{\bar{\triangledown}} D$. By definition,

$$\begin{array}{rl} & (w,x) \mathbin{\bar{\triangledown}} D \\ = & (w,x) \mathbin{\bar{\triangledown}} (x,x) \\ = & \{(w',x) \mid w' \in w \triangledown_\cap x\} \\ = & \{(w',x) \mid \exists w_1, w_2.\ w = w' \circ w_1 \text{ and } x = w' \circ w_2 \text{ and } w' \sharp w_1 \sharp w_2\} \end{array}$$

Thus, to show $(w,x) \in (w,x) \mathbin{\bar{\triangledown}} D$, we must find w_1, w_2 with $w \sharp w_1 \sharp w_2$ and $w = w \circ w_1$ and $x = w \circ w_2$. There is an $e \in E$ such that $w \circ e = w$, and w' such that $x = w \circ w'$. By picking $w_1 = e$ and $w_2 = w'$, we are done.

Contraction: Given $(w,x) \in \bar{W}$, we have to show that $(w,x) \in (w,x) \mathbin{\bar{\triangledown}} (w,x)$. We have $(w,x) \mathbin{\bar{\triangledown}} (w,x) = \{(w',x) \mid w' \in w \triangledown_\cap w\}$ by definition, and $w \in w \triangledown_\cap w$ by the contraction property for $\triangledown_\cap$ (Theorem 4.5), which completes the case.

Weak distribution: Supposing that

$$((w_1,x_1) \mathbin{\bar{\circ}} (w_2,x_2)) \cap ((w_3,x_3) \mathbin{\bar{\triangledown}} (w_4,x_4)) \neq \emptyset$$

we require to find $(z,x) \in \bar{W}$ such that $(w_3,x_3) \in (w_1,x_1) \mathbin{\bar{\circ}} (z,x)$ and $(w_2,x_2) \in (z,x) \mathbin{\bar{\triangledown}} (w_4,x_4)$. By definition of $\bar{\circ}$ and $\bar{\triangledown}$, we have

$$\{(w,x) \mid w = w_1 \circ w_2 \text{ and } x \in x_1 \cup x_2\} \cap \{(w',x_3) \mid w' \in w_3 \triangledown_\cap w_4\} \neq \emptyset\ .$$

Thus, $x_3 = x_4 \in x_1 \cup x_2$ and $(w_1 \circ w_2) \cap (w_3 \triangledown_\cap w_4) \neq \emptyset$. By the weak distribution property for $\triangledown_\cap$ wr.t $\langle W, \circ, E\rangle$ (Theorem 4.5), we get z such that $w_3 = w_1 \circ z$ and $w_2 \in z \triangledown_\cap w_4$. Now, letting $x = x_3$, there is a $w \in W$ such that $x = w_3 \circ w$, so we have $x = z \circ (w_1 \circ w)$, and hence $(z,x) \in \bar{W}$. Then, $x \in x \cup x_1$ by the second part of Lemma 4.8, and so, as required,

$$(w_3,x) \in (w_1,x_1) \mathbin{\bar{\circ}} (z,x) \quad \text{and} \quad (w_2,x) \in (z,x) \mathbin{\bar{\triangledown}} (w_4,x_4)\ . \qquad \square$$

Our final result, stated as Theorem 4.12, is that, if M has the cross-split and the disjointness properties, then our constructed BiBBI-model $\bar{M}$ satisfies *all* the properties of Table 1 except classicality. The following lemma groups together a number of intermediary results used in the proof of this theorem.

Lemma 4.11. *Suppose that $M = \langle W, \circ, E\rangle$ is partial functional and has the cross-split and disjointness properties, and let $\bar{M} = \langle \bar{W}, \bar{\circ}, \bar{E}, \bar{\triangledown}, D\rangle$ be as in Definition 4.9. All of the following hold:*

1. *For all $(w_1,x), (w_2,x) \in \bar{W}$, we have $w_1 \triangledown_\cap w_2 = \{w\}$ for some $w \in W$ (and in the following we typically drop the singleton set brackets).*

   Consequently, $\bar{\triangledown}$ is a partial function on $\bar{W} \times \bar{W}$.

2. *If $(w,x), (w_1 \circ w_2, x) \in \bar{W}$, then*

$$w \sharp w_1 \text{ and } w \sharp w_2 \text{ implies } (w \circ w_1 \circ w_2, x) \in \bar{W}\ .$$

3. *For all $(w,x), (w_1 \circ w_2, x) \in \bar{W}$,*

$$w \triangledown_\cap (w_1 \circ w_2) = (w \triangledown_\cap w_1) \circ (w \triangledown_\cap w_2)\ .$$

Proof. We start with the following preliminary fact, assuming an underlying BBI-model with the cross-split and disjointness properties:

$$\forall w, w_1, w_2 \in W.\, w \sharp w_1 \sharp w_2 \text{ and } w \circ w_1 = w \circ w_2 \text{ implies } w_1, w_2 \in E \tag{1}$$

Indeed, assume that $w \sharp w_1 \sharp w_2$ and $w \circ w_1 = w \circ w_2$. By the cross-split property, there are a, b, c, d such that $w = a \circ b = a \circ c$, $w_1 = c \circ d$, and $w_2 = b \circ d$. Thus, since $w \sharp w_1 \sharp w_2$ by assumption, we have $a \sharp b \sharp c \sharp d \sharp b \sharp d$ and $a \sharp c \sharp c \sharp d \sharp b \sharp d$. Consequently we have $b \sharp b$ and $c \sharp c$ and $d \sharp d$. By the disjointness property, $b, c, d \in E$, which implies that $w_1, w_2 \in E$.

We now prove each part of the statement of the lemma separately.

1. First, we show that $w_1 \triangledown_\cap w_2$ is nonempty. Since $(w_1, x), (w_2, x) \in \bar{W}$, there are w_1' and w_2' such that $x = w_1 \circ w_1' = w_2 \circ w_2'$. By the cross-split property of M, there are $a, b, c, d \in W$ such that

$$w_1 = a \circ b, \quad w_1' = c \circ d, \quad w_2 = a \circ c, \quad \text{and} \quad w_2' = b \circ d\ .$$

   Since $w_2 \sharp w_2'$, we have $a \sharp c \sharp b \sharp d$, and thus in particular $a \sharp b \sharp c$. Since $w_1 = a \circ b$ and $w_2 = a \circ c$, this gives us $a \in w_1 \triangledown_\cap w_2$.

   On the other hand, supposing that $a, a' \in w_1 \triangledown_\cap w_2$, we require to show that $a' = a$. By definition of $\triangledown_\cap$, there are $y_1, y_2, y_1', y_2' \in W$ such that all of the following hold:

$$w_1 = a \circ y_1 = a' \circ y_1', \quad w_2 = a \circ y_2 = a' \circ y_2', \quad a \sharp y_1 \sharp y_2, \quad \text{and} \quad a' \sharp y_1' \sharp y_2'\ .$$

   By cross-split applied to both equalities above, there are $\alpha, \beta_i, \gamma_i, \delta_i \in W$, where $i \in \{1, 2\}$, such that

$$a = \alpha \circ \beta_i, \quad y_i = \gamma_i \circ \delta_i, \quad a' = \alpha \circ \gamma_i, \quad \text{and} \quad y_i' = \beta_i \circ \delta_i\ .$$

   Moreover, injecting the facts above into $a' \sharp y_1' \sharp y_2'$ gives us that $\alpha \sharp \gamma_i \sharp \beta_1 \sharp \delta_1 \sharp \beta_2 \sharp \delta_2$, and hence in particular $\alpha \sharp \beta_1 \sharp \beta_2$. Thus, we can apply fact (1) to obtain $\beta_1, \beta_2 \in E$. Similarly, $a \sharp y_1 \sharp y_2$ yields $\gamma_1, \gamma_2 \in E$. Thus, $a = \alpha = a'$ as required.

   It is straightforward to verify that $\bar{\triangledown}$ is then a partial function on $\bar{W} \times \bar{W}$.

2. Assume that the hypotheses hold, *i.e.*, $w \sharp w_1$ and $w \sharp w_2$ and $(w, x), (w_1 \circ w_2, x) \in \bar{W}$. We have to show that $(w \circ w_1 \circ w_2, x) \in \bar{W}$, *i.e.*, that $x = (w \circ w_1 \circ w_2) \circ z$ for some $z \in W$.

   By assumption, there are $w', w'' \in W$ such that $x = w \circ w' = w_1 \circ w_2 \circ w''$. Then, by the cross-split property of M, there are $a, b, c, d \in W$ such that

$$w_1 \circ w_2 = a \circ b, \quad w'' = c \circ d, \quad w = a \circ c, \quad \text{and} \quad w' = b \circ d\ .$$

   By applying cross-split again to the first equality above, we get $\alpha, \beta, \gamma, \delta \in W$ such that

$$a = \alpha \circ \beta, \quad b = \gamma \circ \delta, \quad w_1 = \alpha \circ \gamma, \quad \text{and} \quad w_2 = \beta \circ \delta\ .$$

   Now, since $w_1 \sharp w$, we have that $(\alpha \circ \gamma) \sharp (\alpha \circ \beta \circ c)$, which implies that $\alpha \sharp \alpha$, and then by disjointness of M we get $\alpha \in E$. Similarly, $w_2 \sharp w$ yields $\beta \in E$. Consequently, $a = \alpha \circ \beta \in E$, which means in particular that $w = c$. Thus we have, as required,

$$x = w \circ w' = a \circ c \circ b \circ d = c \circ (a \circ b) \circ d = w \circ w_1 \circ w_2 \circ d\ .$$

3. First, notice that $(w_1 \circ w_2, x) \in \bar{W}$ implies $(w_1, x), (w_2, x) \in \bar{W}$. Thus, by part 1 of the present lemma, we may write

$$y = w \triangledown_\cap (w_1 \circ w_2), \quad y_1 = w \triangledown_\cap w_1 \quad \text{and} \quad y_2 = w \triangledown_\cap w_2 \ .$$

We require to show that $y_1 \circ y_2 = y$. Since $y = w \triangledown_\cap (w_1 \circ w_2)$, we have $y', y'' \in W$ such that

$$w = y \circ y', \quad w_1 \circ w_2 = y \circ y'' \quad \text{and} \quad y \sharp y' \sharp y'' \ .$$

By applying cross-split to the second equality, there are $a, b, c, d \in W$ such that

$$w_1 = a \circ b, \quad w_2 = c \circ d, \quad y = a \circ c \quad \text{and} \quad y'' = b \circ d \ .$$

Now we have $w = y \circ y' = a \circ (c \circ y')$ and $w_1 = a \circ b$. Furthermore, since $y \sharp y' \sharp y''$, we get $a \sharp c \sharp y' \sharp b \sharp d$, which means in particular that $a \sharp (c \circ y') \sharp b$. Therefore, $a \in w \triangledown_\cap w_1$, which means that $a = y_1$. By a similar argument, $c = y_2$. That is, $y = a \circ c = y_1 \circ y_2$ as required. □

Theorem 4.12. *Given a partial functional* BBI*-model M with the cross-split and disjointness properties, $\bar{M}$ is a* BiBBI*-model with all the properties of Table 1 except classicality.*

Proof. All required properties of $\bar{M}$ apart from unit expansion and associativity is taken care of by Theorem 4.10. We establish each of these two properties separately.

Unit expansion: We require to show that $(w, x) \bar{\triangledown} D \subseteq \{(w, x\}$. By construction,

$$(w, x) \bar{\triangledown} U = (w, x) \bar{\triangledown} (x, x) = \{(y, x) \mid y \in w \triangledown_\cap x\} \ .$$

By Lemma 4.11.1, this is necessarily a singleton set, so it suffices to establish just that $w \in w \triangledown_\cap x$. Since $(w, x) \in \bar{M}$, there is a $w' \in W$ with $x = w \circ w'$. Since there is an $e \in E$ such that $w = w \circ e$ and $w \circ w' = w \circ (w' \circ e)$ (and hence also $w \sharp w' \sharp e$), we have $w \in w \triangledown_\cap (w \circ w') = w \triangledown_\cap x$ as required.

Associativity: We require to show that

$$(w_1, x_1) \bar{\triangledown} ((w_2, x_2) \bar{\triangledown} (w_3, x_3)) = ((w_1, x_1) \bar{\triangledown} (w_2, x_2)) \bar{\triangledown} (w_3, x_3) \ .$$

When $x_i \neq x_j$ for some $i, j \in \{1, 2, 3\}$, both sides of the equation collapse to $\emptyset$ and we are done. Let us thus assume $x_1 = x_2 = x_3 = x$. In that case, by definition of $\bar{\triangledown}$, we require to prove

$$w_1 \triangledown_\cap (w_2 \triangledown_\cap w_3) = (w_1 \triangledown_\cap w_2) \triangledown_\cap w_3) \ .$$

Writing $w = w_2 \triangledown_\cap w_3$, we obtain $w_2', w_3' \in W$ such that

$$w_2 = w \circ w_2', \quad w_3 = w \circ w_3', \quad \text{and} \quad w \sharp w_2' \sharp w_3' \ .$$

Next, writing $w' = w_1 \triangledown_\cap w$, we obtain $w_1', w'' \in W$ such that be such that

$$w_1 = w' \circ w_1', \quad w = w' \circ w'', \quad \text{and} \quad w' \sharp w_1' \sharp w'' \ .$$

Now, writing $y = w_1' \triangledown_\cap w_2'$, we also have $y_1, y_2 \in W$ such that

$$w_1' = y \circ y_1, \quad w_2' = y \circ y_2, \quad \text{and} \quad y \sharp y_1 \sharp y_2 \ .$$

We claim that $w_1 \triangledown_\cap w_2 = y \circ w'$. By Lemma 4.11.1, it suffices to show that $y \circ w' \in w_1 \triangledown_\cap w_2$. Using the equalities above, we have

$$w_1 = w' \circ w_1' = (y \circ w') \circ y_1 \quad \text{and} \quad w_2 = w \circ w_2' = (y \circ w') \circ (w'' \circ y_2)\ .$$

Now, $(y \circ y_1) \sharp y_2$ by the above, and $(y \circ y_1) \sharp (w' \circ w'')$ because $(y \circ y_1) \circ (w' \circ w'') = w_1' \circ w$, which is defined by construction. Also, notice that $(y_2 \circ (w' \circ w''), x) \in \bar{W}$, because $(w_2, x) \in \bar{W}$ and $w_2 = (w' \circ w'') \circ y \circ y_2$. Thus, by Lemma 4.11.2, we get $((y \circ y_1) \circ y_2 \circ (w' \circ w''), x) \in \bar{W}$, which implies that $(y \circ w') \sharp y_1 \sharp (w'' \circ y_2)$. Thus $y \circ w' \in w_1 \triangledown_\cap w_2$ as required.

Now, we calculate

$$\begin{array}{rcll} (w_1 \triangledown_\cap w_2) \triangledown_\cap w_3 & = & (y \circ w') \triangledown_\cap w_3 & \\ & = & (y \triangledown_\cap w_3) \circ (w' \triangledown_\cap w_3) & \text{(by Lemma 4.11.3)} \\ & = & w' \triangledown_\cap w_3 & \\ & = & w' & \\ & = & w_1 \triangledown_\cap (w_2 \triangledown_\cap w_3) & \end{array}$$

For the third equality, observe that $(w \circ w_3') \sharp w_2'$ by assumption, that is, $w_3 \sharp (y \circ y_2)$ and so $y \sharp x_3$. Then by a straightforward calculation (and using Lemma 4.11.1), $y \triangledown_\cap w_3 = e$ for some $e \in E$.

For the fourth equality, observe that $w_3 = w' \circ (w'' \circ w_3)$ by assumption. Similar to the proof of unit expansion above, it is then easy to show that $w' \triangledown_\cap w_3 = w'$. □

5 Completeness of BiBBI

This section presents in detail our proof of completeness for (any variant of) BiBBI, stated earlier as Theorem 3.10. Our approach follows the one previously employed in the literature for BBI [8] and for CBI [4]: we translate (a given variant of) BiBBI to an equivalent presentation as a modal logic, and appeal to the well known completeness result for modal logic due to Sahlqvist (see e.g. [2]). Sahlqvist completeness says, essentially, that when a "normal" modal logic is augmented with axioms of a particular syntactic form, this modal logic is guaranteed to be complete with respect to the class of Kripke frames for the logic in which all the axioms are valid. The main technical challenges are: firstly, to reformulate BiBBI as a set of modal logic axioms of the required Sahlqvist form; and, secondly, to show that these Sahlqvist axioms are derivable when translated back into BiBBI. Unsurprisingly, the weak distribution law presents the greatest difficulty on both counts.

We begin by recalling the standard definitions of validity and provability in normal modal logic over a suitably chosen signature of modalities (called a "modal similarity type" in [2]).

Definition 5.1. A *modal logic formula* is built from propositional variables in $\mathcal{V}$ using the classical connectives, the 0-ary modalities $\top^*$ and $\mathbf{U}$, and the binary modalities $*, -\!\!\circ, \triangledown$ and $\multimapinv$.

Definition 5.2. A *modal frame* is given by $\langle W, \circ, -\!\!\circ, \triangledown, \multimapinv, E, U \rangle$, where $\circ$, $-\!\!\circ$, $\triangledown$, and $\multimapinv$ all have type $W \times W \to \mathcal{P}(W)$, and $E, U \subseteq W$.

A valuation for a modal frame $M = \langle W, \ldots \rangle$ is as usual given by a function $\rho : \mathcal{V} \to \mathcal{P}(W)$. The forcing relation $w \models_\rho A$ is defined by induction on A in the standard way in modal logic, i.e. as for BBI in the case of propositional variables and classical connectives, with the following clauses for the modalities:

$$\begin{array}{rcl}
w \models_\rho \top^* & \Leftrightarrow & w \in E \\
w \models_\rho \mathbf{U} & \Leftrightarrow & w \in U \\
w \models_\rho A * B & \Leftrightarrow & \exists w_1, w_2 \in W.\ w \in w_1 \circ w_2 \text{ and } w_1 \models_\rho A \text{ and } w_2 \models_\rho B \\
w \models_\rho A \multimap B & \Leftrightarrow & \exists w_1, w_2 \in W.\ w \in w_1 \multimap w_2 \text{ and } w_1 \models_\rho A \text{ and } w_2 \models_\rho B \\
w \models_\rho A \triangledown B & \Leftrightarrow & \exists w_1, w_2 \in W.\ w \in w_1 \triangledown w_2 \text{ and } w_1 \models_\rho A \text{ and } w_2 \models_\rho B \\
w \models_\rho A \leftarrow B & \Leftrightarrow & \exists w_1, w_2 \in W.\ w \in w_1 \leftarrow w_2 \text{ and } w_1 \models_\rho A \text{ and } w_2 \models_\rho B
\end{array}$$

As usual, A is *valid* in M iff $w \models_\rho A$ for all $w \in W$ and valuations ρ.

Each of the binary functions $\circ, \multimap, \triangledown, \leftarrow : W \times W \to \mathcal{P}(W)$ in a modal frame can be equivalently seen as a ternary relation over W (which is the standard approach in modal logic). The corresponding modalities are each interpreted as a binary "diamond-type" modality of modal logic. The modal interpretations of $*$ and $\top^*$ are exactly their usual interpretations in BiBBI (see Defn. 2.5), while the modal interpretations of $\mathbf{U}$, $\multimap$, $\triangledown$ and $\leftarrow$ are related to the BiBBI interpretations of $\bot^*$, $-\!\!*$, $\overset{*}{\triangledown}$ and $\backslash^*$ respectively by the use of Boolean negations.

Definition 5.3. The *normal modal logic* $\mathbf{ML}_{\mathrm{BiBBI}}$ for $(\top^*, \mathbf{U}, *, \multimap, \triangledown, \leftarrow)$ is given by extending a standard Hilbert system for classical logic with the following axioms and rules, for all $\otimes \in \{*, \multimap, \triangledown, \leftarrow\}$:

$$\begin{array}{l}
\bot \otimes A \vdash \bot \text{ and } A \otimes \bot \vdash \bot \\
(A \vee B) \otimes C \vdash (A \otimes C) \vee (B \otimes C) \\
A \otimes (B \vee C) \vdash (A \otimes B) \vee (A \otimes C)
\end{array}
\qquad
\frac{A_1 \vdash A_2 \quad B_1 \vdash B_2}{A_1 \otimes B_1 \vdash A_2 \otimes B_2}$$

Next, we recall the Sahlqvist completeness result connecting validity and provability in normal modal logics augmented with suitably well-behaved axioms, called *Sahlqvist formulas.* In fact, we only require so-called "very simple" Sahlqvist formulas for our completeness result.

Definition 5.4. A *very simple Sahlqvist antecedent* (over $(\top^*, \mathbf{U}, *, \multimap, \triangledown, \leftarrow)$) is given by the following grammar:

$$S ::= P \mid \top \mid \bot \mid S \wedge S \mid \top^* \mid \mathbf{U} \mid S * S \mid S \multimap S \mid S \triangledown S \mid S \leftarrow S$$

A *very simple Sahlqvist formula* is an implication $A \vdash B$, where A is a very simple Sahlqvist antecedent and B is any positive modal logic formula (i.e., such that every propositional variable occurs within the scope of an even number of negations).

Theorem 5.5 (Sahlqvist completeness)**.** *Let $\mathcal{A}$ be a set of very simple Sahlqvist formulas. If a modal logic formula is valid in the set of all modal frames satisfying $\mathcal{A}$, then it is provable in* $\mathbf{ML}_{\mathrm{BiBBI}} + \mathcal{A}$*.*

We now define a set of Sahlqvist formulas that collectively capture all variants of BiBBI.

Definition 5.6. For a given variant of BiBBI, define the set $\mathcal{A}_{\mathrm{BiBBI}}$ of very simple Sahlqvist formulas as follows:

$$\begin{array}{rl}
(1) & A \wedge (B * C) \vdash (B \wedge (C \multimap A)) * \top \\
(2) & A \wedge (B \multimap C) \vdash \top \multimap (C \wedge (A * B)) \\
(3) & A \wedge (B \triangledown C) \vdash \top \triangledown (C \wedge (A \multimapinv B)) \\
(4) & A \wedge (B \multimapinv C) \vdash (B \wedge (C \triangledown A)) \multimapinv \top \\
(5) & A * B \vdash B * A \\
(6) & A \triangledown B \vdash B \triangledown A \\
(7) & A * (B * C) \vdash (A * B) * C \\
(8) & A * \top^* \vdash A \text{ and } A \vdash A * \top^* \\
\text{(Associativity)} & A \triangledown (B \triangledown C) \vdash (A \triangledown B) \triangledown C \\
\text{(Unit expansion)} & A \triangledown \mathbf{U} \vdash A \\
\text{(Unit contraction)} & A \vdash A \triangledown \mathbf{U} \\
\text{(Contraction)} & A \vdash A \triangledown A \\
\text{(Weak distribution)} & (A * B) \wedge (C \triangledown D) \vdash (A \wedge ((B \multimapinv D) \multimap C)) * \top \\
\text{(Classicality)} & (A \multimap \mathbf{U}) \multimap \mathbf{U} \vdash A \text{ and } A \vdash (A \multimap \mathbf{U}) \multimap \mathbf{U}
\end{array}$$

where A, B, C, D are considered to be propositional variables, and the named axioms are included in $\mathcal{A}_{\text{BiBBI}}$ if and only if the BiBBI variant includes the corresponding property in Table 1.

Thus, by Theorem 5.5, whenever a modal logic formula is valid in the class of modal frames satisfying $\mathcal{A}_{\text{BiBBI}}$, it is provable in $\mathbf{ML}_{\text{BiBBI}} + \mathcal{A}_{\text{BiBBI}}$. Next, we connect validity in BiBBI to validity in modal frames.

Lemma 5.7. *Let $M = \langle W, \circ, \multimap, \triangledown, \multimapinv, E, U \rangle$ be a modal frame satisfying axioms (1)–(4) of $\mathcal{A}_{\text{BiBBI}}$ in Definition 5.6. Then we have, for any $w, w_1, w_2 \in W$:*

$$w \in w_1 \multimap w_2 \;\Leftrightarrow\; w_2 \in w \circ w_1 \quad \text{and} \quad w \in w_1 \multimapinv w_2 \;\Leftrightarrow\; w_1 \in w_2 \triangledown w$$

Consequently, using the above equivalences and alpha-renaming, the clauses for satisfaction of $\multimap$ and $\multimapinv$ can be rewritten as

$$\begin{array}{lcl}
w \models_\rho A \multimap B & \Leftrightarrow & \exists w', w''.\ w'' \in w \circ w' \text{ and } w' \models_\rho A \text{ and } w'' \models_\rho B \\
w \models_\rho A \multimapinv B & \Leftrightarrow & \exists w', w''.\ w'' \in w' \triangledown w \text{ and } w'' \models_\rho A \text{ and } w' \models_\rho B
\end{array}$$

(That is, the modal interpretations of $A \multimap B$ and $A \multimapinv B$ are exactly the BiBBI *interpretations of $\neg(A \mathrel{-\!\!*} \neg B)$ and $A \mathbin{\backslash^*} \neg B$, respectively.)*

Proof. First, we tackle the bi-implication connecting $\multimap$ and $\circ$.

($\Leftarrow$) Suppose $w_2 \in w \circ w_1$. Define a valuation ρ for M by $\rho(A) = \{w_2\}$, $\rho(B) = \{w\}$, $\rho(C) = \{w_1\}$. Then, by construction, $w_2 \models_\rho A \wedge (B * C)$. Since axiom (1) is valid in M by assumption, $w_2 \models_\rho (B \wedge (C \multimap A)) * \top$. That is, $w_2 \in w' \circ w''$ for some w', w'' such that $w' \models_\rho B \wedge (C \multimap A)$. Since $w' \models_\rho B$, we have $w' = w$ and thus $w \models_\rho C \multimap A$, which means exactly that $w \in w_1 \multimap w_2$.

($\Rightarrow$) Suppose $w \in w_1 \multimap w_2$. Define a valuation for M by $\rho(A) = \{w\}$, $\rho(B) = \{w_1\}$, $\rho(C) = \{w_2\}$. By construction, $w \models_\rho A \wedge (B \multimap C)$. Since axiom (2) is valid in M by assumption, $w \models_\rho \top \multimap (C \wedge (A * B))$. Thus $w' \models_\rho C \wedge (A * B)$ for some w'. Since $w' \models_\rho C$, we have $w' = w_2$ and thus $w_2 \models_\rho A * B$, i.e. $w_2 \in w \circ w_1$ as required.

Next, we similarly establish the bi-implication connecting $\multimapinv$ and $\triangledown$.

($\Leftarrow$) Suppose $w_1 \in w_2 \triangledown w$. Define a valuation for M by $\rho(A) = \{w_1\}$, $\rho(B) = \{w_2\}$, $\rho(C) = \{w\}$. By construction, $w_1 \models_\rho A \wedge (B \triangledown C)$. Since axiom (3) is valid in M, we have

$w \models_\rho \top \triangledown (C \wedge (A \leftarrow B))$. Thus $w' \models_\rho C \wedge (A \leftarrow B)$ for some w', and since $w' \models_\rho C$ we have $w' = w$ and thus $w \models_\rho A \leftarrow B$. That is, $w \in w_1 \leftarrow w_2$ as required.

($\Rightarrow$) Suppose $w \in w_1 \leftarrow w_2$. Define a valuation for M by $\rho(A) = \{w\}$, $\rho(B) = \{w_1\}$, $\rho(C) = \{w_2\}$. By construction, $w \models_\rho A \wedge (B \leftarrow C)$. Since axiom (4) is valid in M, we have $w \models_\rho (B \wedge (C \triangledown A) \leftarrow \top$. Thus $w' \models_\rho B \wedge (C \triangledown A)$ for some w', and since $w' \models_\rho B$ we have $w' = w_1$ and thus $w_1 \models_\rho C \triangledown A$. This means exactly that $w_1 \in w_2 \triangledown w$, as required. □

In the following, given a modal frame $M = \langle W, \circ, \multimap, \triangledown, \leftarrow, E, U\rangle$, we shall write $\ulcorner M \urcorner$ for the restricted tuple $\langle W, \circ, E, \triangledown, U\rangle$.

Lemma 5.8. *Let $M = \langle W, \circ, \multimap, \triangledown, \leftarrow, E, U\rangle$ be a modal frame satisfying the set $\mathcal{A}_{\mathrm{BiBBI}}$ of axioms corresponding to a* BiBBI *variant, as given by Definition 5.6. Then $\ulcorner M \urcorner$ is a* BiBBI-*model for that variant.*

Proof. First of all, it is easy to verify that $\ulcorner M \urcorner$ is a basic BiBBI-model, since it satisfies axioms (5)–(8) in Definition 5.6. It then suffices to show that if an optional Sahlqvist axiom from Definition 5.6 is valid in M, then M satisfies the corresponding frame property in Table 1 (and thus, as an immediate consequence, $\ulcorner M \urcorner$ does too). This is a straightforward verification for all of the axioms except weak distribution and classicality. Since classicality is covered in [4], we only consider the case of weak distribution here.

Thus, assume the weak distribution axiom of Definition 5.6 is valid in M and suppose that $(x_1 \circ x_2) \cap (y_1 \triangledown y_2) \neq \emptyset$. That is, we have $z \in (x_1 \circ x_2) \cap (y_1 \triangledown y_2)$ for some $z \in W$. We require to find a $w \in W$ such that $y_1 \in x_1 \circ w$ and $x_2 \in w \triangledown y_2$. Define a valuation ρ for M by $\rho(A) = \{x_1\}$, $\rho(B) = \{x_2\}$, $\rho(C) = \{y_1\}$ and $\rho(D) = \{y_2\}$. By construction, $z \models_\rho (A * B) \wedge (C \triangledown D)$. Since the weak distribution axiom is valid in M, we have $z \models_\rho (A \wedge ((B \leftarrow D) \multimap C)) * \top$. That is, for some z' we have $z' \models_\rho A \wedge ((B \leftarrow D) \multimap C)$. Since $z' \models_\rho A$, we have $z' = x_1$ and thus $x_1 \models_\rho (B \leftarrow D) \multimap C$. As M satisfies axioms (1)–(4) by assumption, we can apply Lemma 5.7 to obtain

$$\exists w, w'.\ w' \in x_1 \circ w \text{ and } w \models_\rho B \leftarrow D \text{ and } w' \models_\rho C$$

As $w \models_\rho C$, we have $y_1 \in x_1 \circ w$. Using Lemma 5.7 and commutativity of $\triangledown$ (forced by the validity of axiom (6) in M), we obtain from $w \models_\rho B \leftarrow D$ that

$$\exists w', w''.\ w'' \in w \triangledown w' \text{ and } w'' \models_\rho B \text{ and } w' \models_\rho D$$

i.e., $x_2 \in w \triangledown y_2$ as required. This completes the proof. □

We now formally define the obvious formula translations between modal logic and BiBBI.

Definition 5.9. We define a translation $t(-)$ from BiBBI-formulas to modal logic formulas, and a symmetric translation $u(-)$ in the opposite direction, as follows:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
t(\phi) & = & \phi & u(\phi) & = & \phi \\
t(\bot^*) & = & \neg\mathbf{U} & u(\mathbf{U}) & = & \neg\bot^* \\
t(\neg A) & = & \neg t(A) & u(\neg A) & = & \neg u(A) \\
t(A \mathbin{?} B) & = & t(A) \mathbin{?} t(B) & u(A \mathbin{?} B) & = & u(A) \mathbin{?} u(B) \\
t(A \mathrel{-\!\!*} B) & = & \neg(t(A) \multimap \neg B) & u(A \multimap B) & = & \neg(u(A) \mathrel{-\!\!*} \neg u(B)) \\
t(A \overset{*}{\triangledown} B) & = & \neg(\neg t(A) \triangledown \neg t(B)) & u(A \triangledown B) & = & \neg(\neg u(A) \overset{*}{\triangledown} \neg u(B)) \\
t(A \setminus^* B) & = & t(A) \leftarrow \neg t(B) & u(A \leftarrow B) & = & u(A) \setminus^* \neg u(B)
\end{array}$$

where $\phi \in \{P, \top, \bot, \top^*\}$ and $? \in \{\wedge, \vee, \rightarrow, *\}$.

Lemma 5.10. *Suppose a* BiBBI*-formula A is valid in some variant of* BiBBI*. Then $t(A)$ is valid in the class of modal frames satisfying the corresponding Sahlqvist axioms $\mathcal{A}_{\mathrm{BiBBI}}$ given by Definition 5.6.*

Proof. Let $M = \langle W, \circ, \multimap, \triangledown, \leftarrowtriangle, E, U \rangle$ be a modal frame satisfying the axioms $\mathcal{A}_{\mathrm{BiBBI}}$. We require to show that $t(A)$ is valid in M. By Lemma 5.8, $\ulcorner M \urcorner$ is a BiBBI-model for the variant of BiBBI determined by $\mathcal{A}_{\mathrm{BiBBI}}$, and thus A is valid in $\ulcorner M \urcorner$. Note that since the set of underlying worlds W is identical in M and $\ulcorner M \urcorner$, any valuation for M is a valuation for $\ulcorner M \urcorner$ and vice versa. It thus suffices to show the following bi-implication, for all $w \in W$ and valuations ρ:

$$w \models_\rho A \text{ (in } \ulcorner M \urcorner) \;\Leftrightarrow\; w \models_\rho t(A) \text{ (in } M)$$

(The satisfaction relations for BiBBI (Defn. 3.2) and for modal logic (Defn. 5.2) agree where the logical connectives of both languages, BiBBI and modal logic, overlap; therefore, we do not disambiguate explicitly between them.) We proceed by structural induction on the formula A.

Cases $A = P, \top, \bot, \top^*$. Immediate since $t(A) = A$.

Case $A = \bot^*$. We have $w \models_\rho \bot^* \Leftrightarrow w \notin U \Leftrightarrow w \models_\rho \neg\mathbf{U}$, as required.

Cases $A = A_1 \vee A_2, A_1 \wedge A_2, A_1 \to A_2, A_1 * A_2$. These cases are all similar, so we just show the case $A = A_1 * A_2$. In that case, using the induction hypotheses for A_1 and A_2, we have

$$\begin{array}{lll}
 & w \models_\rho A_1 * A_2 & \\
\Leftrightarrow & \exists w_1, w_2.\ w \in w_1 \circ w_2 \text{ and } w \models_\rho A_1 \text{ and } w \models_\rho A_2 & \\
\Leftrightarrow & \exists w_1, w_2.\ w \in w_1 \circ w_2 \text{ and } w \models_\rho t(A_1) \text{ and } w \models_\rho t(A_2) & \text{(by ind. hyp)} \\
\Leftrightarrow & w \models_\rho t(A_1) * t(A_2) & \\
\Leftrightarrow & w \models_\rho t(A_1 * A_2) &
\end{array}$$

Case $A = A_1 \mathrel{-\!\!*} A_2$. We have

$$\begin{array}{lll}
 & w \models_\rho A_1 \mathrel{-\!\!*} A_2 & \\
\Leftrightarrow & \forall w', w''.\ w'' \in w \circ w' \text{ and } w' \models_\rho A_1 \text{ implies } w'' \models_\rho A_2 & \\
\Leftrightarrow & \forall w', w''.\ w'' \in w \circ w' \text{ and } w' \models_\rho t(A_1) \text{ implies } w'' \models_\rho t(A_2) & \text{(by ind. hyp)} \\
\Leftrightarrow & \neg\exists w', w''.\ w'' \in w \circ w' \text{ and } w' \models_\rho t(A_1) \text{ and } w'' \models_\rho \neg t(A_2) & \\
\Leftrightarrow & w \models_\rho \neg(t(A_1) \multimap \neg t(A_2)) & \text{(Lemma 5.7)} \\
\Leftrightarrow & w \models_\rho t(A_1 \mathrel{-\!\!*} A_2) &
\end{array}$$

Case $A = A_1 \overset{*}{\vee} A_2$. We have

$$\begin{array}{lll}
 & w \models_\rho A_1 \overset{*}{\vee} A_2 & \\
\Leftrightarrow & \forall w_1, w_2.\ w \in w_1 \triangledown w_2 \text{ implies } w_1 \models_\rho A_1 \text{ or } w_2 \models_\rho A_2 & \\
\Leftrightarrow & \forall w_1, w_2.\ w \in w_1 \triangledown w_2 \text{ implies } w_1 \models_\rho t(A_1) \text{ or } w_2 \models_\rho t(A_2) & \text{(by ind. hyp)} \\
\Leftrightarrow & \neg\exists w_1, w_2.\ w \in w_1 \triangledown w_2 \text{ and } w_1 \models_\rho \neg t(A_1) \text{ and } w_2 \models_\rho \neg t(A_2) & \\
\Leftrightarrow & w \models_\rho \neg(\neg t(A_1) \triangledown \neg t(A_2)) & \\
\Leftrightarrow & w \models_\rho t(A_1 \overset{*}{\vee} A_2) &
\end{array}$$

Case $A = A_1 \setminus^{*} A_2$. We have

$$\begin{array}{lll}
 & w \models_\rho A_1 \setminus^{*} A_2 & \\
\Leftrightarrow & \exists w', w''.\ w'' \in w' \triangledown w \text{ and } w'' \models_\rho A_1 \text{ and } w' \not\models_\rho A_2 & \\
\Leftrightarrow & \exists w', w''.\ w'' \in w' \triangledown w \text{ and } w'' \models_\rho t(A_1) \text{ and } w' \not\models_\rho t(A_2) & \text{(by ind. hyp)} \\
\Leftrightarrow & \exists w', w''.\ w'' \in w' \triangledown w \text{ and } w'' \models_\rho t(A_1) \text{ and } w' \models_\rho \neg t(A_2) & \\
\Leftrightarrow & w \models_\rho t(A_1) \leftarrowtriangle \neg t(A_2) & \text{(Lemma 5.7)} \\
\Leftrightarrow & w \models_\rho t(A_1 \setminus^{*} A_2) &
\end{array}$$

This completes all cases. □

Using Lemma 5.10 and Theorem 5.5, we have that validity of a BiBBI-formula A in some BiBBI variant implies provability of $t(A)$ in $\mathbf{ML}_{\text{BiBBI}} + \mathcal{A}_{\text{BiBBI}}$. It just remains to connect modal logic provability to BiBBI-provability.

Lemma 5.11. *If a modal logic formula B is provable in $\mathbf{ML}_{\text{BiBBI}} + \mathcal{A}_{\text{BiBBI}}$, then $u(B)$ is provable in the corresponding variant of* BiBBI.

Proof. We have to show that all the axioms and rules of normal modal logic (see Definition 5.3) and all the $\mathcal{A}_{\text{BiBBI}}$ axioms (see Definition 5.6) are derivable in the appropriate variant of BiBBI under the translation $u(-)$. For the most part this is a straightforward (if tedious) exercise; the main exceptions are the residuation axioms and the weak distribution axiom. We take for granted that BiBBI-provability is closed under the usual classical principles, in particular modus ponens and the De Morgan laws for Boolean negation $\neg$.

First, we treat the normal modal logic axioms that govern the interaction of the binary modalities with $\bot$.

Cases $A * \bot \vdash \bot$ and $\bot * A \vdash \bot$. We just show the first case, as the other is similar. We have to show that $u(A) * \bot \vdash u(A)$ is provable in the basic proof system for BiBBI (see Definition 3.3). Write $B = u(A)$. We have $\bot \vdash B \mathrel{-\!\!*} B$ provable, so by the residuation rules for $*$ and $\mathrel{-\!\!*}$, we have that $\bot * B \vdash B$ is provable and thus $B * \bot \vdash B$ is provable by commutativity of $*$.

Cases $A \multimap \bot \vdash \bot$ and $\bot \multimap A \vdash \bot$. For the first case, we have to show that $\neg(u(A) \mathrel{-\!\!*} \top) \vdash \bot$ is provable. Write $B = u(A)$. Certainly $\top * B \vdash \top$ is provable, so by residuation $B \mathrel{-\!\!*} \top$ is provable, and thus $\neg(B \mathrel{-\!\!*} \top) \vdash \bot$ follows by contraposition.

For the second case, we have to show that $\neg(\bot \mathrel{-\!\!*} u(A)) \vdash \bot$ is provable. Write $B = u(A)$. We already know that $\top * \bot \vdash \bot$ is provable (see the case above). Thus, since $\bot \vdash B$ is provable, so is $\top * \bot \vdash B$. By residuation, $\bot \mathrel{-\!\!*} B$ is then provable, and thus we get $\neg(\bot \mathrel{-\!\!*} B) \vdash \bot$ as required by contraposition.

Cases $A \triangledown \bot \vdash \bot$ and $\bot \triangledown A \vdash \bot$. We just show the first case, as the other is similar. We have to show that $\neg(\neg u(A) \mathbin{\overset{*}{\vee}} \top) \vdash \bot$ is provable. Write $B = \neg u(A)$. Trivially, $\top \mathbin{\backslash^{*}} B \vdash \top$ is provable. Thus, by the residuation rules for $\overset{*}{\vee}$ and $\backslash^{*}$, so is $B \mathbin{\overset{*}{\vee}} \top$. We then get $\neg(B \mathbin{\overset{*}{\vee}} \top) \vdash \bot$ by contraposition.

Cases $A \mathbin{\triangleleft\!\!-} \bot \vdash \bot$ and $\bot \mathbin{\triangleleft\!\!-} A \vdash \bot$. For the first case, we have to show $u(A) \mathbin{\backslash^{*}} \top \vdash \bot$ is provable. Write $B = u(A)$. We trivially have $B \mathbin{\backslash^{*}} \bot \vdash \top$ provable, so by residuation $B \vdash \bot \mathbin{\overset{*}{\vee}} \top$ is provable. By commutativity of $\overset{*}{\vee}$ (provided by basic provability in BiBBI) we then get $B \vdash \top \mathbin{\overset{*}{\vee}} \bot$, and so by residuation $B \mathbin{\backslash^{*}} \top \vdash \bot$ as required.

For the second case, we have to show $\bot \mathbin{\backslash^{*}} \neg u(A) \vdash \bot$ is provable. Writing $B = \neg u(A)$, we trivially have $\bot \vdash B \mathbin{\overset{*}{\vee}} \bot$ provable. Thus, by residuation, $\bot \mathbin{\backslash^{*}} B \vdash \bot$ is provable.

Next, we treat the normal modal logic axioms showing that the binary modalities distribute over $\vee$ (in both argument positions).

Case $(A \vee B) * C \vdash (A * C) \vee (B * C)$. We need to show that the following is provable:

$$(u(A) \vee u(B)) * u(C) \vdash (u(A) * u(C)) \vee (u(B) * u(C))$$

Write $A' = u(A)$, $B' = u(B)$, $C' = u(C)$, and $D = (A' * C') \vee (B' * C')$. Trivially, we have $A' * C' \vdash D$ provable, so by residuation $A' \vdash C' \mathrel{-\!\!*} D$ is provable. By the same token, $B' \vdash C' \mathrel{-\!\!*} D$ is also provable. Thus, $A' \vee B' \vdash C' \mathrel{-\!\!*} D$ is provable, and so $(A' \vee B') * C' \vdash D$ follows by residuation as required.

Case $A * (B \vee C) \vdash (A * B) \vee (A * C)$. Similar to the previous case, additionally using the commutativity of $*$.

Case $(A \vee B) \multimap C \vdash (A \multimap C) \vee (B \multimap C)$. We need to show that the following is provable:

$$\neg((u(A) \vee u(B)) \mathrel{-\!\!*} \neg u(C)) \vdash \neg(u(A) \mathrel{-\!\!*} \neg u(C)) \vee \neg(u(B) \mathrel{-\!\!*} \neg u(C))$$

Write $A' = u(A)$, $B' = u(B)$, $C' = \neg u(C)$, $D = A' \mathrel{-\!\!*} C'$ and $E = B' \mathrel{-\!\!*} C'$. First, note that $D \wedge E \vdash A' \mathrel{-\!\!*} C'$ is trivially provable. Thus, using residuation and commutativity of $*$, we have $A' \vdash (D \wedge E) \mathrel{-\!\!*} C'$ provable. By the same token, $B' \vdash (D \wedge E) \mathrel{-\!\!*} C'$ is also provable. Thus, $A' \vee B' \vdash (D \wedge E) \mathrel{-\!\!*} C'$ is provable. By residuation and $*$-commutativity, we then obtain $D \wedge E \vdash (A' \vee B') \mathrel{-\!\!*} C'$. We then get $\neg((A' \vee B') \mathrel{-\!\!*} C') \vdash \neg D \vee \neg E$ by contraposition as required.

Case $A \multimap (B \vee C) \vdash (A \multimap B) \vee (A \multimap C)$. We require to prove the following:

$$\neg(u(A) \mathrel{-\!\!*} (\neg u(B) \wedge \neg u(C))) \vdash \neg(u(A) \mathrel{-\!\!*} \neg u(B)) \vee \neg(u(A) \mathrel{-\!\!*} \neg u(C))$$

Write $A' = u(A)$, $B' = \neg u(B)$, $C' = \neg u(C)$, $D = A' \mathrel{-\!\!*} B'$ and $E = A' \mathrel{-\!\!*} C'$. We trivially have $D \wedge E \vdash A' \mathrel{-\!\!*} B'$. By residuation we get $(D \wedge E) * A' \vdash B'$, and similarly $(D \wedge E) * A' \vdash C'$, so we can get $(D \wedge E) * A' \vdash B' \wedge C'$. Then, by residuation, we have $D \wedge E \vdash A' \mathrel{-\!\!*} (B' \wedge C')$, and so by contraposition we can obtain $\neg(A' \mathrel{-\!\!*} (B' \wedge C')) \vdash \neg D \vee \neg E$, as required.

Case $(A \vee B) \triangledown C \vdash (A \triangledown C) \vee (B \triangledown C)$. We need to show that the following is provable:

$$\neg((\neg u(A) \wedge \neg u(B)) \overset{*}{\triangledown} \neg u(C)) \vdash \neg(\neg u(A) \overset{*}{\triangledown} \neg u(C)) \vee \neg(\neg u(B) \overset{*}{\triangledown} \neg u(C))$$

Write $A' = \neg u(A)$, $B' = \neg u(B)$, $C' = \neg u(C)$, $D = A' \overset{*}{\triangledown} C'$ and $E = B' \overset{*}{\triangledown} C'$. We can trivially prove $D \wedge E \vdash A' \overset{*}{\triangledown} C'$. By commutativity of $\overset{*}{\triangledown}$ and the residuation rules, we can get $(D \wedge E) \backslash^{*} C' \vdash A'$. By a similar line of reasoning, we can also prove $(D \wedge E) \backslash^{*} C' \vdash B'$, so we can prove $(D \wedge E) \backslash^{*} C' \vdash A' \wedge B'$. Using residuation and commutativity again, we then obtain $D \wedge E \vdash (A' \wedge B') \overset{*}{\triangledown} C'$. Finally, by contraposition we get $\neg((A' \wedge B') \overset{*}{\triangledown} C') \vdash \neg D \vee \neg E$, as required.

Case $A \triangledown (B \vee C) \vdash (A \triangledown B) \vee (A \triangledown C)$. Similar to the previous case.

Case $(A \vee B) \leftarrow C \vdash (A \leftarrow C) \vee (B \leftarrow C)$. We need to prove:

$$(u(A) \vee u(B)) \backslash^{*} \neg u(C) \vdash (u(A) \backslash^{*} \neg u(C)) \vee (u(B) \backslash^{*} \neg u(C))$$

Write $A' = u(A)$, $B' = u(B)$, $C' = \neg u(C)$, $D = A' \backslash^{*} C'$ and $E = B' \backslash^{*} C'$. We easily have $A' \backslash^{*} C' \vdash D \vee E$ provable, so by residuation we get $A' \vdash C' \overset{*}{\triangledown} (D \vee E)$. Similarly, we can prove $B' \vdash C' \overset{*}{\triangledown} (D \vee E)$, and so we obtain $A' \vee B' \vdash C' \overset{*}{\triangledown} (D \vee E)$. By residuation, we then get $(A' \vee B') \backslash^{*} C' \vdash D \vee E$ as required.

Case $A \leftarrow (B \vee C) \vdash (A \leftarrow B) \vee (A \leftarrow C)$. We need to prove:

$$u(A) \backslash^{*} (\neg u(B) \wedge \neg u(C)) \vdash (u(A) \backslash^{*} \neg u(B)) \vee (u(A) \backslash^{*} \neg u(C))$$

Write $A' = u(A)$, $B' = \neg u(B)$, $C' = \neg u(C)$, $D = A' \backslash^{*} B'$ and $E = A' \backslash^{*} C'$. We can trivially prove $A' \backslash^{*} B' \vdash D \vee E$. By residuation and commutativity of $\overset{*}{\triangledown}$ we can obtain $A' \backslash^{*} (D \vee E) \vdash B'$. By a similar argument we can also obtain $A' \backslash^{*} (D \vee E) \vdash C'$, so we have $A' \backslash^{*} (D \vee E) \vdash B' \wedge C'$. Then, applying residuation and commutativity again, we can derive $A' \backslash^{*} (B' \wedge C') \vdash D \vee E$ as required.

Next on our list are the monotonicity rules for the binary modalities in normal modal logic.

Case: monotonicity rule for $*$*.* We have to show that the following rule is admissible:

$$\frac{u(A_1) \vdash u(B_1) \qquad u(A_2) \vdash u(B_2)}{u(A_1) * u(A_2) \vdash u(B_1) * u(B_2)}$$

We are immediately done since the above is an instance of the usual monotonicity rule for $*$ in BBI (cf. Definition 2.2).

Case: monotonicity rule for $\multimap$*.* We have to show that the following rule is admissible:

$$\frac{u(A_1) \vdash u(B_1) \qquad u(A_2) \vdash u(B_2)}{\neg(u(A_1) \mathrel{-\!\!*} \neg u(A_2)) \vdash \neg(u(B_1) \mathrel{-\!\!*} \neg u(B_2))}$$

Write $A'_1 = u(A_1)$, $A'_2 = u(A_2)$, $B'_1 = u(B_1)$ and $B'_2 = u(B_2)$. Trivially, $B'_1 \mathrel{-\!\!*} \neg B'_2 \vdash B'_1 \mathrel{-\!\!*} \neg B'_2$ is provable. By applying residuation and contraposition, we can derive $B'_2 \vdash \neg((B'_1 \mathrel{-\!\!*} \neg B'_2) * B'_1)$. Since $A'_2 \vdash B'_2$ is provable by assumption, we can obtain $A'_2 \vdash \neg((B'_1 \mathrel{-\!\!*} \neg B'_2) * B'_1)$ by transitivity of implication. Using residuation, commutativity of $*$ and contraposition, this is equivalent to $B'_1 \vdash (B'_1 \mathrel{-\!\!*} \neg B'_2) \mathrel{-\!\!*} \neg A'_2$. Since $A'_1 \vdash B'_1$ is provable by assumption, we have by transitivity $A'_1 \vdash (B'_1 \mathrel{-\!\!*} \neg B'_2) \mathrel{-\!\!*} \neg A'_2$. Finally, using residuation, commutativity, and contraposition again, this is equivalent to the required $\neg(A'_1 \mathrel{-\!\!*} \neg A'_2) \vdash \neg(B'_1 \mathrel{-\!\!*} \neg B'_2)$.

Case: monotonicity rule for $\triangledown$*.* We have to show that the following rule is admissible:

$$\frac{u(A_1) \vdash u(B_1) \qquad u(A_2) \vdash u(B_2)}{\neg(\neg u(A_1) \mathbin{\overset{*}{\vee}} \neg u(A_2)) \vdash \neg(\neg u(B_1) \mathbin{\overset{*}{\vee}} \neg u(B_2))}$$

Using contraposition, we can rewrite this rule in the following equivalent form:

$$\frac{\neg u(B_1) \vdash \neg u(A_1) \qquad \neg u(B_2) \vdash \neg u(A_2)}{\neg u(B_1) \mathbin{\overset{*}{\vee}} \neg u(B_2) \vdash \neg u(A_1) \mathbin{\overset{*}{\vee}} \neg u(A_2)}$$

which is an instance of the monotonicity rule for $\overset{*}{\vee}$ in the basic BiBBI system (see Defn. 3.3).

Case: monotonicity rule for $\prec\!-$*.* We have to show that the following rule is admissible:

$$\frac{u(A_1) \vdash u(B_1) \qquad u(A_2) \vdash u(B_2)}{u(A_1) \mathbin{\setminus^{*}} \neg u(A_2) \vdash u(B_1) \mathbin{\setminus^{*}} \neg u(B_2)}$$

Write $A'_1 = u(A_1)$, $A'_2 = u(A_2)$, $B'_1 = u(B_1)$ and $B'_2 = u(B_2)'$. Trivially, $B'_1 \mathbin{\setminus^{*}} \neg B'_2 \vdash B'_1 \mathbin{\setminus^{*}} \neg B'_2$ is provable. By residuation, $B'_1 \vdash \neg B'_2 \mathbin{\overset{*}{\vee}} (B'_1 \mathbin{\setminus^{*}} \neg B'_2)$ is then provable. Since $A'_1 \vdash B'_1$ is provable by assumption, we obtain $A'_1 \vdash \neg B'_2 \mathbin{\overset{*}{\vee}} (B'_1 \mathbin{\setminus^{*}} \neg B'_2)$ by transitivity. Using residuation, contraposition and commutativity of $\overset{*}{\vee}$, this is equivalent to $B'_2 \vdash \neg(A'_1 \mathbin{\setminus^{*}} (B_1 \mathbin{\setminus^{*}} \neg B'_2))$. Since $A'_2 \vdash B'_2$ is provable by assumption, we then have $A'_2 \vdash \neg(A'_1 \mathbin{\setminus^{*}} (B_1 \mathbin{\setminus^{*}} \neg B'_2))$ by transitivity. Using residuation, commutativity and contraposition once more, this is equivalent to the required $A'_1 \mathbin{\setminus^{*}} \neg A'_2 \vdash B'_1 \mathbin{\setminus^{*}} \neg B'_2$.

This completes the cases for all the principles of normal modal logic $\mathbf{ML}_{\mathrm{BiBBI}}$. Next, we must examine the Sahlqvist axioms given by Definition 5.6. We begin by treating the axioms (1)–(8), bearing in mind that we can still only assume the proof principles of basic BiBBI (cf. Definition 3.3).

Case: axiom (1)*,* $A \wedge (B * C) \vdash (B \wedge (C \multimap A)) * \top$*.* We need to prove:

$$u(A) \wedge (u(B) * u(C)) \vdash (u(B) \wedge \neg(u(C) \mathrel{-\!\!*} \neg u(A))) * \top$$

Write $A' = u(A)$, $B' = u(B)$, $C' = u(C)$, $D = (B' \wedge \neg(C' \mathrel{-\!\!*} \neg A')) * \top$ and $E = C' \mathrel{-\!\!*} (A' \to D)$. We trivially have $\neg A' \vdash A' \to D$ a classical tautology. Since $C' * (C' \mathrel{-\!\!*} \neg A') \vdash \neg A'$ is easily provable in BBI, we obtain $C' * (C' \mathrel{-\!\!*} \neg A') \vdash A' \to D$ by transitivity. Using $*$-commutativity, residuation and contraposition, this is equivalent to $\neg(C' \mathrel{-\!\!*} (A' \to D)) \vdash \neg(C' \mathrel{-\!\!*} \neg A')$, i.e. $\neg E \vdash \neg(C' \mathrel{-\!\!*} \neg A')$. Thus, using the monotonicity rule for $*$ and the fact that $C' \vdash \top$ is provable, we can obtain $(B' \wedge \neg E) * C' \vdash (B' \wedge \neg(C' \mathrel{-\!\!*} \neg A')) * \top$, that is, $(B' \wedge \neg E) * C' \vdash D$. Since $D \vdash A' \to D$ is a tautology, we have by transitivity $(B' \wedge \neg E) * C' \vdash A' \to D$ and then by residuation $B' \wedge \neg E \vdash C' \mathrel{-\!\!*} (A' \to D)$, i.e., $B' \wedge \neg E \vdash E$. This is classically equivalent to $B' \vdash E$, i.e. $B' \vdash C' \mathrel{-\!\!*} (A' \to D)$. By residuation, this then yields $A' \wedge (B' * C') \vdash D$ as required.

Case: axiom (2)*,* $A \wedge (B \multimap C) \vdash \top \multimap (C \wedge (A * B))$*.* We need to prove:

$$u(A) \wedge \neg(u(B) \mathrel{-\!\!*} \neg u(C)) \vdash \neg(\top \mathrel{-\!\!*} (\neg u(C) \vee \neg(u(A) * u(B))))$$

Write $A' = u(A)$, $B' = u(B)$, $C' = \neg u(C)$, $D = \top \mathrel{-\!\!*} (C' \vee \neg(A' * B'))$ and $E = (A' \wedge D) * B'$. We easily have $A' \wedge D \vdash A'$ provable, so by the monotonicity rule for $*$ we can obtain $(A' \wedge D) * B' \vdash A' * B'$, i.e. $E \vdash A' * B'$. Using contraposition, we get $\neg(A' * B') \vdash \neg E$. Consequently, we can derive $C' \vee \neg(A' * B') \vdash C' \vee \neg E$. As $D * B' \vdash C' \vee \neg(A' * B')$ is easily derivable in BBI, we obtain $D * B' \vdash C' \vee \neg E$ by transitivity. By residuation this is equivalent to $D \vdash B' \mathrel{-\!\!*} (C' \vee \neg E)$, and since $A' \wedge D \vdash D$ is trivially derivable we get $A' \wedge D \vdash B' \mathrel{-\!\!*} (C' \vee \neg E)$ by transitivity. Thus, by residuation, we get $(A' \wedge D) * B \vdash C' \vee \neg E$, that is, $E \vdash C' \vee \neg E$. This is classically equivalent to $E \vdash C'$, i.e., $(A' \wedge D) * B \vdash C'$. Using residuation and contraposition, this rearranges to $A' \wedge \neg(B' \mathrel{-\!\!*} C') \vdash \neg D$ as required.

Case: axiom (3)*,* $A \wedge (B \triangledown C) \vdash \top \triangledown (C \wedge (A \leftarrow B))$*.* We need to prove:

$$u(A) \wedge \neg(\neg u(B) \overset{*}{\triangledown} \neg u(C)) \vdash \neg(\bot \overset{*}{\triangledown} (\neg u(C) \vee \neg(u(A) \mathbin{\backslash^{*}} \neg u(B))))$$

Write $A' = u(A)$, $B' = \neg u(B)$, $C' = \neg u(C)$, $D = \bot \overset{*}{\triangledown} (C' \vee \neg(A' \mathbin{\backslash^{*}} B'))$ and $E = (A' \wedge D) \mathbin{\backslash^{*}} B'$. We trivially can prove $A' \mathbin{\backslash^{*}} B' \vdash A' \mathbin{\backslash^{*}} B'$, which by residuation is equivalent to $A' \vdash B' \overset{*}{\triangledown} (A' \mathbin{\backslash^{*}} B')$. As $A' \wedge D \vdash A'$ is trivially provable, we have $A' \wedge D \vdash B' \overset{*}{\triangledown} (A' \mathbin{\backslash^{*}} B')$ by transitivity and thus $\neg(A' \mathbin{\backslash^{*}} B') \vdash \neg((A' \wedge D) \mathbin{\backslash^{*}} B')$ by residuation and contraposition. That is, we have $\neg(A' \mathbin{\backslash^{*}} B') \vdash \neg E$. Consequently, we can derive $C' \vee \neg(A' \mathbin{\backslash^{*}} B') \vdash C' \vee \neg E$. Since $\bot \vdash B'$ is trivially provable, we obtain $\bot \overset{*}{\triangledown} (C' \vee \neg(A' \mathbin{\backslash^{*}} B')) \vdash B' \overset{*}{\triangledown} (C' \vee \neg E)$ by the monotonicity rule for $\overset{*}{\triangledown}$, that is, $D \vdash B' \overset{*}{\triangledown} (C' \vee \neg E)$. Since $A' \wedge D \vdash D$ is trivially provable, we have $A' \wedge D \vdash B' \overset{*}{\triangledown} (C' \vee \neg E)$ by transitivity. By residuation, this is equivalent to $(A' \wedge D) \overset{*}{\triangledown} B' \vdash C' \vee \neg E$, that is, $E \vdash C' \vee \neg E$. This is classically equivalent to $E \vdash C'$, i.e., $(A' \wedge D) \mathbin{\backslash^{*}} B' \vdash C'$. This can then be rewritten to the required $A' \wedge \neg(B' \overset{*}{\triangledown} C') \vdash \neg D$ using residuation and contraposition.

Case: axiom (4)*,* $A \wedge (B \leftarrow C) \vdash (B \wedge (C \triangledown A)) \leftarrow \top$*.* We need to prove:

$$u(A) \wedge (u(B) \mathbin{\backslash^{*}} \neg u(C)) \vdash (u(B) \wedge \neg(\neg u(C) \overset{*}{\triangledown} \neg u(A))) \mathbin{\backslash^{*}} \bot$$

Write $A' = u(A)$, $B' = u(B)$, $C' = \neg u(C)$, $D = (B' \wedge \neg(C' \overset{*}{\triangledown} \neg A')) \mathbin{\backslash^{*}} \bot$ and $E = C' \overset{*}{\triangledown} (D \vee \neg A')$. We can trivially prove both $\neg A' \vdash D \vee \neg A'$ and $C' \vdash C'$, so by applying the rule for monotonicity of $\overset{*}{\triangledown}$ we can prove $C' \overset{*}{\triangledown} \neg A' \vdash C' \overset{*}{\triangledown} (D \vee \neg A')$, i.e., $C' \overset{*}{\triangledown} \neg A' \vdash E$. By contraposition, we obtain $\neg E \vdash \neg(C' \overset{*}{\triangledown} \neg A')$. Thus we can derive $B' \wedge \neg E \vdash B' \wedge \neg(C' \overset{*}{\triangledown} \neg A')$. Now, since $\neg C' \vdash \neg\bot$ is a classical tautology, we can apply the translated monotonicity rule for $\leftarrow$ (see the relevant case above) to obtain $(B' \wedge \neg E) \mathbin{\backslash^{*}} C' \vdash (B' \wedge \neg(C' \overset{*}{\triangledown} \neg A')) \mathbin{\backslash^{*}} \bot$, that is, $(B' \wedge \neg E) \mathbin{\backslash^{*}} C' \vdash D$. Since $D \vdash D \vee \neg A'$ is derivable, we then obtain $(B' \wedge \neg E) \mathbin{\backslash^{*}} C' \vdash D \vee \neg A'$ by transitivity. Residuation then yields $B' \wedge \neg E \vdash C' \overset{*}{\triangledown} (D \vee \neg A')$, that is, $B' \wedge \neg E \vdash E$. This is classically equivalent to $B' \vdash E$, i.e., $B' \vdash C' \mathbin{\overset{*}{\vee}} (D \vee \neg A')$. Using residuation and contraposition, this rearranges to $A' \wedge (B' \mathbin{\backslash^{*}} C') \vdash D$ as required.

Case: axiom (5)*,* $A * B \vdash B * A$*.* Immediate: The translation of this axiom is an instance of BBI's axiom of commutativity of $*$.

Case: axiom (6)*,* $A \triangledown B \vdash B \triangledown A$*.* We have to prove:

$$\neg(\neg u(A) \mathbin{\overset{*}{\vee}} \neg u(B)) \vdash \neg(\neg u(B) \mathbin{\overset{*}{\vee}} \neg u(A))$$

By contraposition, this is equivalent to $\neg u(B) \mathbin{\overset{*}{\vee}} \neg u(A) \vdash \neg u(A) \mathbin{\overset{*}{\vee}} \neg u(B)$, which is just an instance of BiBBI's axiom of commutativity of $\overset{*}{\vee}$.

Case: axiom (7)*,* $A * (B * C) \vdash (A * B) * C$*.* Immediate: The translation of this axiom is an instance of BBI's axiom of associativity of $*$.

Case: axiom (8)*,* $A * \top^* \vdash A$ *and* $A \vdash A * \top^*$*.* Immediate: The translations of these axioms are both instances of the corresponding unit law axioms in BBI.

Now it just remains to treat the optional axioms in Definition 5.6. For each such axiom, we may assume that BiBBI provability includes the relevant axiom from Table 1.

Case: (Associativity), $A \triangledown (B \triangledown C) \vdash (A \triangledown B) \triangledown C$*.* We require to show that the following is provable:

$$\neg(\neg u(A) \mathbin{\overset{*}{\vee}} (\neg u(B) \mathbin{\overset{*}{\vee}} \neg u(C))) \vdash \neg((\neg u(A) \mathbin{\overset{*}{\vee}} \neg u(B)) \mathbin{\overset{*}{\vee}} \neg u(C))$$

Write $A' = \neg u(A)$, $B' = \neg u(B)$, and $C' = \neg u(C)$. We have $C' \mathbin{\overset{*}{\vee}} (B' \mathbin{\overset{*}{\vee}} A') \vdash (C' \mathbin{\overset{*}{\vee}} B') \mathbin{\overset{*}{\vee}} A'$ provable (since we are in BiBBI with associativity). Using commutativity of $\overset{*}{\vee}$, this can be rewritten to $(A' \mathbin{\overset{*}{\vee}} B') \mathbin{\overset{*}{\vee}} C' \vdash A' \mathbin{\overset{*}{\vee}} (B' \mathbin{\overset{*}{\vee}} C')$. By contraposition, we then obtain the required $\neg(A' \mathbin{\overset{*}{\vee}} (B' \mathbin{\overset{*}{\vee}} C')) \vdash (\neg(A' \mathbin{\overset{*}{\vee}} B') \mathbin{\overset{*}{\vee}} C')$.

Case: (Unit expansion), $A \triangledown \boldsymbol{U} \vdash A$*.* We require to prove the following:

$$\neg(\neg u(A) \mathbin{\overset{*}{\vee}} \bot^*) \vdash u(A)$$

Write $B = u(A)$. Since the unit expansion axiom of BiBBI is available, $\neg B \vdash \neg B \mathbin{\overset{*}{\vee}} \bot^*$ is provable, and thus $\neg(\neg B \mathbin{\overset{*}{\vee}} \bot^*) \vdash B$ is provable by contraposition.

Case: (Unit contraction), $A \vdash A \triangledown \boldsymbol{U}$*.* We require to prove the following:

$$u(A) \vdash \neg(\neg u(A) \mathbin{\overset{*}{\vee}} \bot^*)$$

Write $B = u(A)$. Since the unit contraction axiom of BiBBI is available, $\neg B \mathbin{\overset{*}{\vee}} \bot^* \vdash \neg B$ is provable, and thus $B \vdash \neg(\neg B \mathbin{\overset{*}{\vee}} \bot^*)$ is provable by contraposition.

Case: (Contraction), $A \vdash A \triangledown A$*.* Similar to the case of unit contraction above.

Case: (Classicality), $(A \multimap \boldsymbol{U}) \multimap \boldsymbol{U} \vdash A$ *and* $A \vdash (A \multimap \boldsymbol{U}) \multimap \boldsymbol{U}$*.* This case is covered in detail in [4] (see Proposition 4.3.2 in that paper).

Case: (Weak distribution), $(A * B) \wedge (C \triangledown D) \vdash (A \wedge ((B \leftarrow D) \multimap C)) * \top$*.* We require to check that the following is provable:

$$(u(A) * u(B)) \wedge \neg(\neg u(C) \mathbin{\overset{*}{\vee}} \neg u(D)) \vdash (u(A) \wedge \neg((u(B) \mathbin{\backslash^{*}} \neg u(D)) \mathbin{-\!\!*} \neg u(C)) * \top$$

Write $A' = u(A)$, $B' = u(B)$, $C' = \neg u(C)$, $D' = \neg u(D)$, $E = (B' \mathbin{\backslash^{*}} D') \mathbin{-\!\!*} C'$, $F = (A' \wedge \neg E) * \top$ and $G = B' \mathbin{-\!\!*} ((C' \mathbin{\overset{*}{\vee}} D') \vee F)$. We require to prove $(A' * B') \wedge \neg(C' \mathbin{\overset{*}{\vee}} D') \vdash F$.

First, note that $B' \vdash (B' \mathbin{\setminus^{*}} D') \mathbin{\overset{*}{\vee}} D'$ is easily provable using residuation and commutativity of $\overset{*}{\vee}$. Hence, by the rule for monotonicity of $*$, we have $E * B' \vdash E * ((B' \mathbin{\setminus^{*}} D') \mathbin{\overset{*}{\vee}} D')$. Now, as $E * ((B' \mathbin{\setminus^{*}} D') \mathbin{\overset{*}{\vee}} D') \vdash (E * (B' \mathbin{\setminus^{*}} D')) \mathbin{\overset{*}{\vee}} D'$ is an instance of the BiBBI weak distribution axiom, $E * B' \vdash (E * (B' \mathbin{\setminus^{*}} D')) \mathbin{\overset{*}{\vee}} D'$ is provable by transitivity. Using $\overset{*}{\vee}$-commutativity and residuation, this rearranges to $(E * B') \mathbin{\setminus^{*}} D' \vdash E * (B' \mathbin{\setminus^{*}} D')$. Since $E = (B' \mathbin{\setminus^{*}} D') \mathrel{-\!\!*} C'$, we easily have $E * (B' \mathbin{\setminus^{*}} D') \vdash C'$ using residuation, and so $(E * B') \mathbin{\setminus^{*}} D' \vdash C'$ is provable by transitivity. Using residuation and commutativity again, this rewrites to $E * B' \vdash C' \mathbin{\overset{*}{\vee}} D'$. By weakening for $\vee$, we can then prove $E * B' \vdash (C' \mathbin{\overset{*}{\vee}} D') \vee F$, which by residuation is equivalent to $E \vdash B' \mathrel{-\!\!*} ((C' \mathbin{\overset{*}{\vee}} D') \vee F)$, that is, $E \vdash G$. By contraposition and standard principles for $\wedge$, we can derive $A' \wedge \neg G \vdash A' \wedge \neg E$. Since $B' \vdash \top$ is trivially derivable, we can apply the monotonicity rule for $*$ to obtain $(A' \wedge \neg G) * B' \vdash (A' \wedge \neg E) * \top$, that is, $(A' \wedge \neg G) * B' \vdash F$. By $\vee$-weakening, $(A' \wedge \neg G) * B') \vdash (C' \mathbin{\overset{*}{\vee}} D') \vee F$ is also derivable. By residuation, we can then prove $A' \wedge \neg G \vdash B' \mathrel{-\!\!*} ((C' \mathbin{\overset{*}{\vee}} D') \vee F)$, that is, $A' \wedge \neg G \vdash G$, which is classically equivalent to $A' \vdash G$. Expanding the definition of G, this is equal to $A' \vdash B' \mathrel{-\!\!*} ((C' \mathbin{\overset{*}{\vee}} D') \vee F)$, which can be rewritten using residuation and contraposition as $(A' * B') \wedge \neg(C' \mathbin{\overset{*}{\vee}} D') \vdash F$. This completes this case, and the entire proof. □

Lemma 5.12. *If $u(t(A))$ is provable in (some variant of)* BiBBI *then so is A.*

Proof. (Sketch) The formula $u(t(A))$ differs from A only in that some subformulas C of A are replaced by their double negations $\neg\neg C$, which clearly does not affect provability in a classical system. The result can be proven formally by a straightforward structural induction on A. □

We now have everything needed to prove our completeness result (stated as Theorem 3.10): every formula valid in a given variant of BiBBI is provable in that variant.

Proof of Theorem 3.10. Suppose A is valid in some BiBBI variant. By Lemma 5.10, $t(A)$ is then valid in the class of modal frames satisfying the Sahlqvist formulas $\mathcal{A}_{\mathrm{BiBBI}}$ given by Definition 5.6. Using the Sahlqvist completeness theorem (Theorem 5.5), $t(A)$ is provable in $\mathbf{ML}_{\mathrm{BiBBI}} + \mathcal{A}_{\mathrm{BiBBI}}$. Thus, by Lemma 5.11, $u(t(A))$ is provable in the corresponding variant of BiBBI. By Lemma 5.12, A is then provable in this BiBBI variant as required. □

6 Conclusions

In this paper, we formulate and investigate a *bi-intuitionistic* bunched logic BiBBI, where the multiplicative disjunction $\overset{*}{\vee}$ and its adjoint co-implication $\setminus^{*}$ have equal status to the usual multiplicative conjunction $*$ and its adjoint implication $-\!\!*$; and, furthermore, all the multiplicatives are treated essentially intuitionistically. From the point of view of linear and modal logic, BiBBI can be seen as a free combination of classical logic with Hyland and de Paiva's (multiplicative) FILL [16]. From the bunched logic perspective, BiBBI is an extension of BBI, and CBI is a special case of BiBBI obtained by imposing a "classicality" condition (cf. Proposition 3.7).

We have formulated a Kripke frame semantics for BiBBI in which various logical axioms of FILL have natural semantic correspondents as first-order conditions on BiBBI-models (cf. Table 1). We provide a completeness proof for this semantics, based on the Sahlvist completeness theorem for modal logic, and moreover our proof provides completeness for *any* variant of BiBBI given by a choice of logical principles from Table 1.

We also investigate BiBBI-models obeying the *weak distribution* law,

$$A * (B \mathbin{\overset{*}{\vee}} C) \vdash (A * B) \mathbin{\overset{*}{\vee}} C,$$

which is of special importance in connecting the $(*, -\!\!*, \top^*)$ fragment of BiBBI to its otherwise disjoint $(\overset{*}{\vee}, \backslash^{*}, \bot^*)$ fragment. Specifically, we find that heap-like models of BiBBI, as used in separation logic, can be obtained by interpreting $\overset{*}{\vee}$ using natural notions of heap *intersection*. Perhaps unfortunately, this comes at the expense of either the associativity of $\overset{*}{\vee}$ or of the unit law $A \overset{*}{\vee} \bot^* \equiv A$. However, we provide a general construction that, given a sufficiently well-behaved BBI-model, yields a more complex BiBBI-models in which these laws do hold, based on pairing every world in the original model with a "larger" one (Theorem 4.12).

As for proof theory, we remark that it ought to be straightforward to construct a cut-eliminating *display calculus* (cf. [1, 3]) for any variant of BiBBI by combining a display calculus for classical logic with the display calculus for the multiplicative fragment of FILL given by Clouston et al [9].

We hope, but do not yet know, that there might exist applications of BiBBI to program verification based on separation logic, beyond those already provided by BBI. As in linear logic, it seems more difficult to reason intuitively using multiplicative disjunction than using multiplicative conjunction. However, the existence of natural heap models of BiBBI gives us some cause for cautious optimism; we hope to explore this direction further in future work.

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